Path Algebras in Quantum Causal Theory

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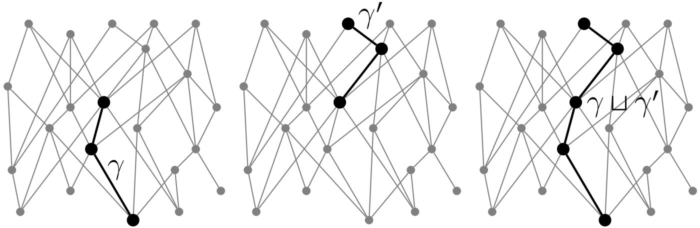
Introduction

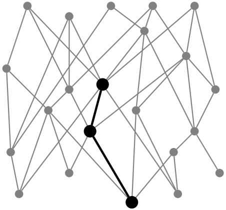
- Path algebras are generated by directed paths in a graph:



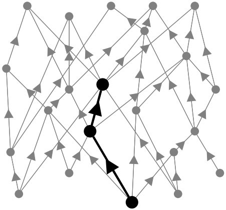
- Physics: direction encodes causal structure.
- This talk: path algebras and quantum dynamics.

Notation and Conventions

- Distinguished terms are blue: path algebra.
- Math symbols in equations are red: $i\hbar\frac{\partial\psi^-}{\partial t} = \mathbf{H}\psi^-$.
- Math symbols in figures are black: see last slide.
- My own material is green: co-relative histories.
- Graphs are acyclic directed "up the page:"



means

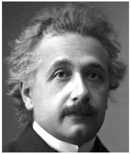


Personalities


Newton


Cauchy


Riemann


Einstein


Schrödinger


Wheeler

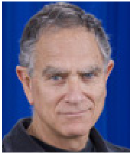

Feynman


Grothendieck


Hawking

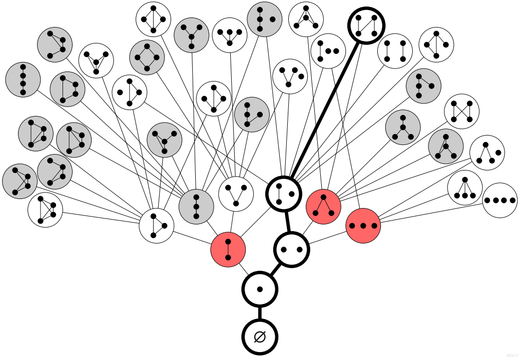

Sorkin


Connes


Malament

Where We're Headed

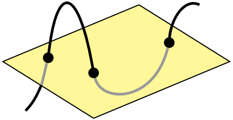
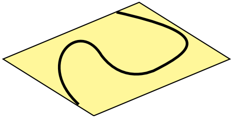
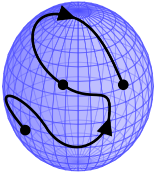
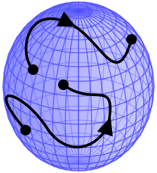
We'll look at paths in a "causal multiverse." Here's a small part:



Path algebra leads to causal Schrödinger-type equation [1]:

$$\psi^{-}_{R;\theta}(r) = \theta(r) \sum_{r^- \prec r} \psi^{-}_{R;\theta}(r^-).$$

Partially Defined Operations

	Defined	Not Defined
• Subtraction (e.g., in $\mathbb{N}$):	$5-3$	$3-5$
• Division (e.g., in $\mathbb{Z}$):	$6/2$	$7/2$
• Matrix Multiplication:	$\begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$
• Proper Intersection: (algebraic geometry)		
• Path Concatenation: (algebraic topology)		

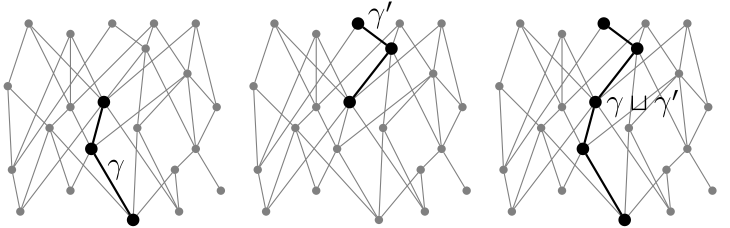
• Many other examples!

Temptation: "Fix" Partially Defined Operations

- Subtraction: form Grothendieck group of $\mathbb{N}$, which is $\mathbb{Z}$.
- Division: localize $\mathbb{Z}$ to get $\mathbb{Q}$. (Leaves out 0!)
- Matrix multiplication: restrict to $n \times n$ to get ring M_n.
- Proper intersection: impose adequate equivalence to get Chow ring.
- Path concatenation: apply homotopy theory to get fundamental group.
- Other examples: derived categories, etc.
- **Questions:** ***should*** **a given partially defined operation be "fixed?" If so, at what level? Can "good behavior" be reconciled with preservation of information?**

Path Algebras

- Generated by directed paths in a graph G:



- Partially defined operation: concatenation $(\gamma, \gamma') \mapsto \gamma \sqcup \gamma'$.
- Coefficients: any ring A.
- Multiplication: $\Big(\sum_{\gamma} a_\gamma \gamma\Big)\Big(\sum_{\gamma'} a_{\gamma'} \gamma'\Big) := \sum_{\gamma \sqcup \gamma' \exists} a_\gamma a_{\gamma'} \gamma \sqcup \gamma'$.
- Appeared independently in multiple fields; e.g., see [2].

Path Algebras as Semicategory Algebras

- Space $\Gamma(G)$ of directed paths in G is a semicategory, or non-unital category.
- Two views regarding $\Gamma(G)$:
  1. Group-like *object;* paths are *elements*.

group	monoid	semigroup	semicategory

  2. Category-like *family;* paths are *morphisms*.

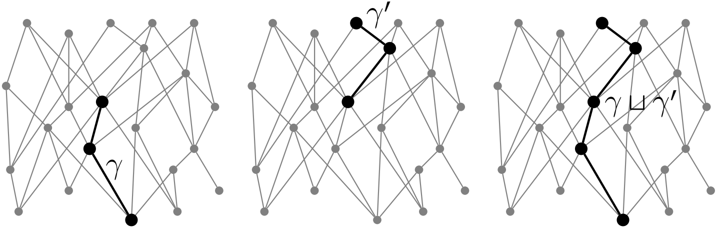
category	semicategory

- $\Gamma(G)$: "generalized group algebra."
- Rich source of examples in noncommutative algebra and geometry.
- Aside: are we too committed to categories?

Connes

Path Algebras and Causal Structure

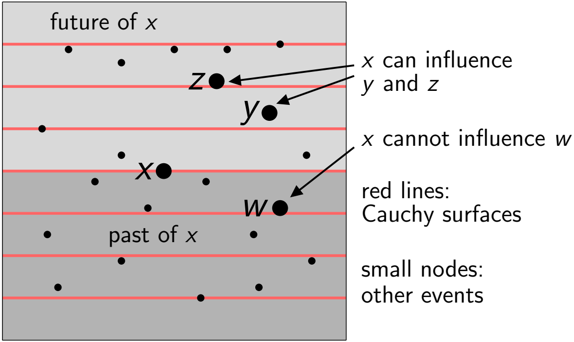
- Physics heuristic: causal influence flows from γ into γ':



- Definition has *physical meaning*: influence flows; histories meet.
- "Fixing" partially defined operation $\sqcup$ would spoil this interpretation!

Causal Structure in Newtonian Physics

- Newton: time is universal; an event can influence any event in its future.

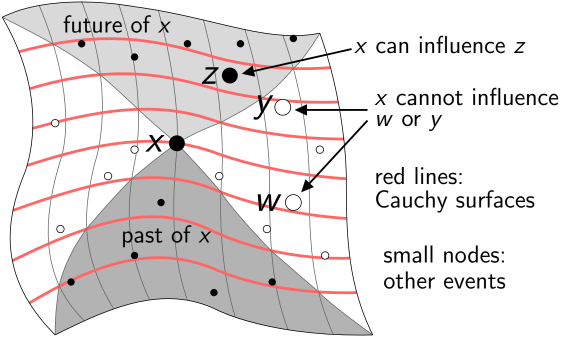


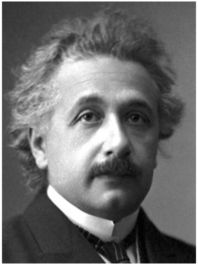
Newton

- Dynamical law: $\frac{d\mathbf{p}}{dt} = \mathbf{F}.$

Causal Structure in Relativity

- Einstein: causal structure is determined by spacetime geometry.



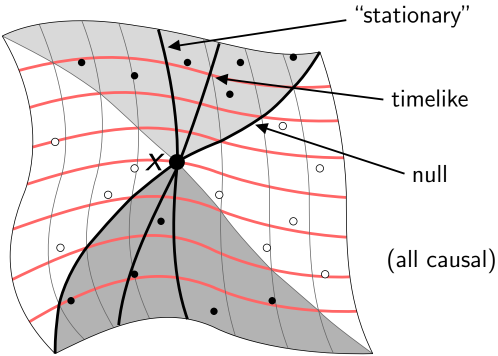


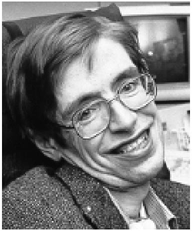
Einstein

- Dynamical law: $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$.

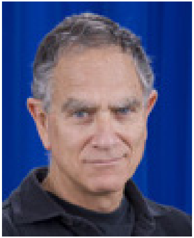
Recovery of Metric from Causal Structure?

- What about the converse? To what extent does causal structure determine relativistic spacetime geometry?





Hawking

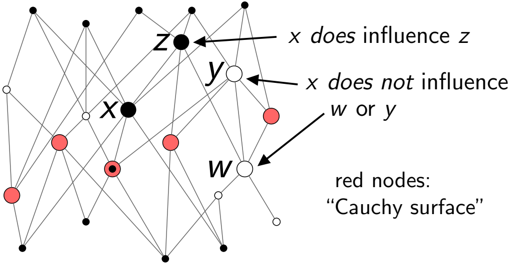


Malament

- Hawking-Malament [3], [4]: causal structure determines geometry up to a conformal factor.

Causal Set Theory

- Sorkin: in discrete setting, counting measure plays role of conformal factor [5]. Hence, "order plus number equals geometry."



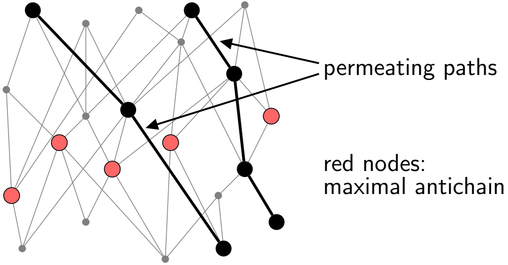
- Special case of causal metric hypothesis [1].
- Riemann: "in a discrete manifold, the ground of metric relations is given in the notion of it."
- Aside: domains; e.g., Martin-Panangaden [6].

Sorkin

Riemann

Cauchy Surfaces in Causal Set Theory

- Cauchy surface: filters information.
  - Newtonian: constant-time section.
  - Relativity: spacelike hypersurface.
  - Causal set theory: maximal antichain.
  - (Aside: Cauchy, Dirichlet, Neumann...)
- Problem: maximal antichains are permeable!

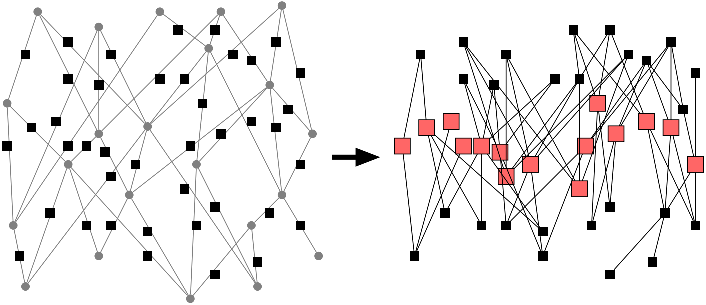


Cauchy

- Complicates "$3 + 1$ approach" to dynamics.

Relation Space

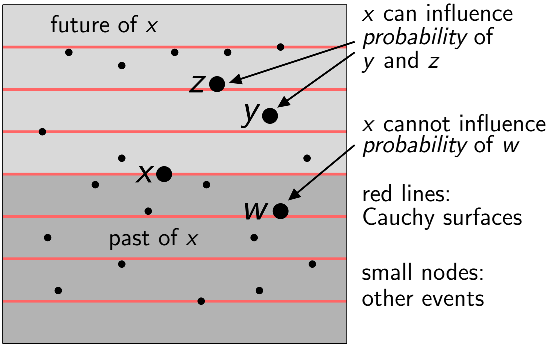
- Solution to permeability: move to relation space!



- Elements in relation space are *relations* between events.
- Mathematically: relation space is a line digraph.
- Analogous to a morphism category.
- Maximal antichains in relation space are impermeable [1].
- Later: Grothendieck's relative viewpoint.

Causal Structure in Quantum Physics

- Schrödinger: probability amplitudes encode likelihood of events.

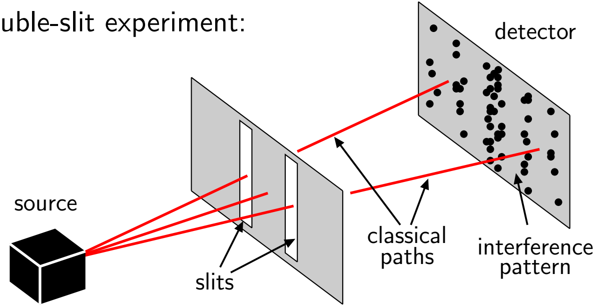


Schrödinger

- Causal theory remains subtle and controversial! (Measurement problem, Bell, no-cloning, etc.)
- Dynamical law: $i\hbar\frac{\partial\psi^-}{\partial t} = \mathbf{H}\psi^-$.

Histories in Quantum Theory

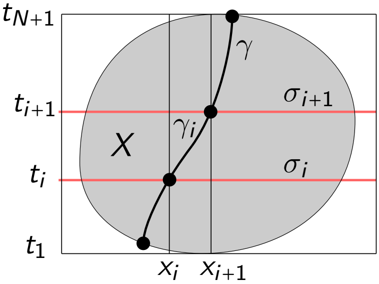
- Double-slit experiment:



- Each path to the detector represents a particle history.
- Particles emitted individually build up interference pattern!
- Somehow all histories are relevant!

Sum over Histories

- Feynman: all histories contribute to probability amplitude ψ, with phases given by action $\mathcal{S}$ for Lagrangian $\mathcal{L}$ [7].



X spacetime region

γ path, γ_i "path increment"

σ_i, σ_{i+1} Cauchy surfaces

Feynman

- Path integral for probability amplitude:

$$\psi(X;\mathcal{L}) := \lim_{|\Delta|\to 0} \int_{\sigma_N} \dots \int_{\sigma_1} \int_{\sigma_0} C \cdot \exp\left(\sum_{i=1}^{N} \frac{i}{\hbar} \mathcal{S}(\gamma_i) \right) dx_0 dx_1 \dots dx_N$$

Feynman Recovers Schrödinger

- Shorthand for path integral:

$$\psi(X;\mathcal{L}) := \lim_{|\Delta|\to 0}\int_{\mathbf{x}}\psi(\Delta;\mathbf{x};\mathcal{L})d\mathbf{x}.$$

- Past wave function:

$$\psi^{-}(x',t') := \lim_{|\Delta^{-}|\to 0}\int_{\mathbf{x}^{-}}\psi^{-}(\Delta^{-};\mathbf{x}^{-};\mathcal{L})d\mathbf{x}^{-}.$$

- Approximate recursion:

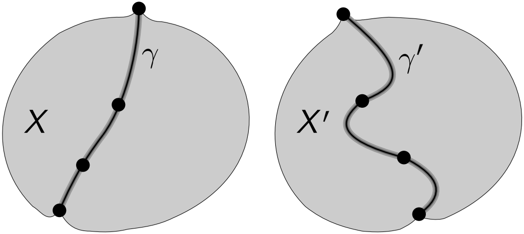
$$\psi^{-}(x'',t'') \approx \int_{\sigma'}\psi^{-}(x',t')\exp\Big(\frac{i}{\hbar}\mathcal{S}(\delta\gamma)\Big)dx'.$$

- Take limit:

$$i\hbar\frac{\partial\psi^{-}}{\partial t} = \mathbf{H}\psi^{-}.$$

Background Independence

- Wheeler: "matter tells spacetime how to curve; spacetime tells matter how to move" [8].

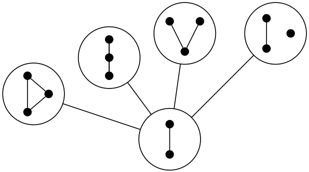


Wheeler

- Different γ and γ' imply different X and X'.
- Generalization: *different histories mean different universes!*
- Sum over histories becomes sum over universes!

Co-Relative Histories

- Grothendieck: "not objects; *relationships between objects.*" (relative viewpoint)
- Here, "objects" are histories (universes).
- Co-relative history: relationship between histories [1].
- Four simple examples, discrete case:

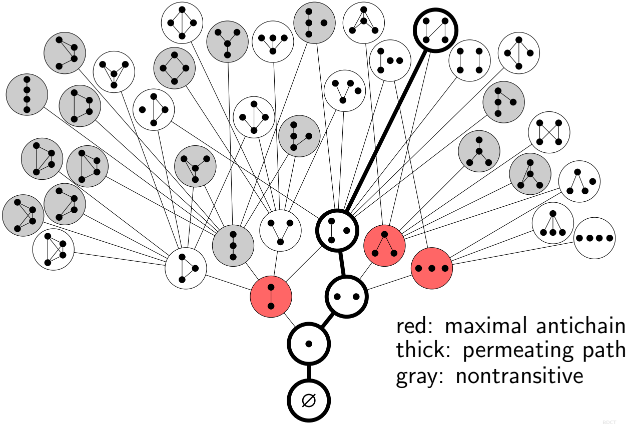


Grothendieck

- Note iteration of structure; a graph of graphs!
- Quantization is iteration of structure in quantum causal theory! (Aside: categorification.)

Kinematic Schemes

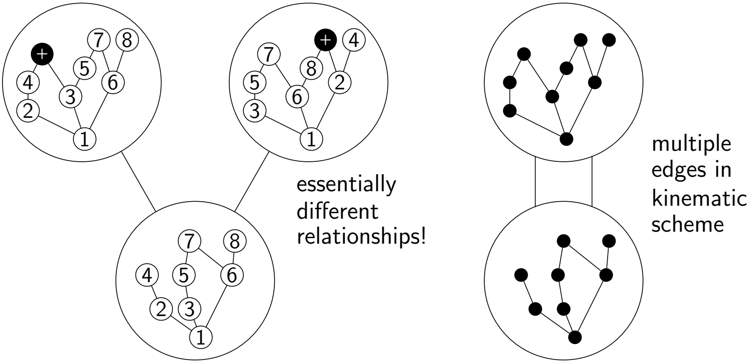
- Kinematic scheme: special "causal multiverse" built from universes and co-relative histories. Here's a portion of one:



- Related: Sorkin's sequential growth dynamics [9].

Multidirected Structure

- Are the simplest co-relative histories between a fixed pair of universes unique? **No!**
- Example due to Brendan McKay [10]:



- Significance: kinematic schemes are generally multidirected.

Causal Schrödinger-Type Equations I

- Recall Schrödinger's equation:

$$i\hbar\frac{\partial\psi^-}{\partial t} = \mathbf{H}\psi^-.$$

- **Goal:** derive discrete causal analogue:

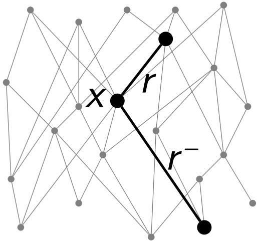
$$\psi^-_{R;\theta}(r) = \theta(r)\sum_{r^-\prec r}\psi^-_{R;\theta}(r^-).$$

- **Motivations:**
  - Quantum spacetime and quantum gravity.
  - Less ambitious: "random lattice field theory."
  - Quantum circuits and quantum computing.
  - Intrinsic order-theoretic and graph-theoretic interest.

Causal Schrödinger-Type Equations II

Setup to derive $\psi^{-}_{R;\theta}(r) = \theta(r) \sum_{r^- \prec r} \psi^{-}_{R;\theta}(r^-)$:

- R relation space over a graph G.
- r^-, r consecutive elements of R sharing vertex x (i.e., $r^- \prec r$.)

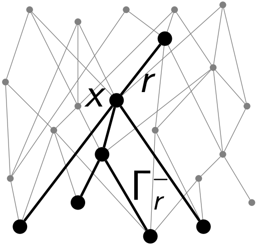


- Note: r^-, r could represent co-relative histories, relations between spacetime events (shown here), part of a quantum circuit, morphisms of some type...

Causal Schrödinger-Type Equations III

Setup to derive $\psi_{R;\theta}^{-}(r) = \theta(r) \sum_{r^{-} \prec r} \psi_{R;\theta}^{-}(r^{-})$ continued:

- $\theta : \Gamma(G) \to A$ phase map into ring A.
- Γ_r^{-} space of maximal paths terminating at x:



- Note: r extends any $\gamma \in \Gamma_r^{-}$.

Causal Schrödinger-Type Equations IV

Past path functional and past wave function:

- Past path functional encodes "all information flowing into r:"

$$\Psi^-_{R;\theta}(r) := \sum_{\gamma\in\Gamma^-_r} \theta(\gamma)\gamma.$$

- Past wave function given by "evaluating" $\Psi^-_{R;\theta}(r)$:

$$\psi^-_{R;\theta}(r) := \sum_{\gamma\in\Gamma^-_r} \theta(\gamma).$$

- Ψ^- is a "path algebra level precursor" to ψ^-.

Causal Schrödinger-Type Equations V

Recursion formula and result:

- Replacing r with r^- in $\Psi^-_{R;\theta}$ leads to recursion:

$$\Psi^-_{R;\theta}(r) = \Big(\sum_{r^- \prec r} \Psi^-_{R;\theta}(r^-)\Big)\theta(r)r.$$

- "Evaluating" yields desired causal Schrödinger-type equation:

$$\psi^-_{R;\theta}(r) = \theta(r)\sum_{r^- \prec r} \psi^-_{R;\theta}(r^-).$$

Concluding Remarks

- Above derivation works for any (acyclic directed) graph G and ring A.
- "Acyclic" can be relaxed ("closed timelike curves.")
- Most interesting: G is a kinematic scheme.
- Related: Isham [11], Raptis [12], Baez [13], etc.
- **Central question: what is the phase map θ?**
  - Involves "Lagrangian" or "action."
  - Related: "discrete Einstein-Hilbert actions." (Dowker-Benincasa [14], etc.)
  - Involves arithmetic, finite groups, Galois theory.

THANKS!

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