Twisted Semicategory Algebras

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Introduction. A *twisted semicategory algebra* is a special type of algebra whose multiplicative structure is defined by modifying the structure of a *semicategory,* also called a *nonunital category,* in a particular way. Twisted semicategory algebras may be viewed as generalized group, monoid, or semigroup algebras. From this perspective, the morphisms of a small semicategory S are treated as elements of a group-like object, and composition of morphisms is treated as a partially-defined algebraic operation on this object. This operation supplies the multiplicative structure of a semicategory algebra $R[S]$ for any ring R. A *twist* is a special modification of this structure, defined in terms of a map $\Theta : S \times S \to R^*$, where R^* is the multiplicative group of invertible elements of R^*. The resulting twisted semicategory algebra is denoted by $R[S;\Theta]$. If G is a group, then the familiar group algebra $R[G]$ may be viewed as the twisted semicategory algebra $R[S_G;\mathbf{1}]$, where S_G is the small category with one object, whose morphisms correspond to the elements of G, with group operation given by composition, and where $\mathbf{1}$ is the constant map $\mathrm{Mor}(S_G)\times\mathrm{Mor}(S_G) \to R^*$ sending each ordered pair of morphisms in S_G to 1_R.

In this project, we will explore relationships among the following:

1. **Semicategory cohomology theories, applied to S.** Group cohomology is the prototypical example. Monoid and semigroup cohomology theories have also been explored to some extent. As one progressively decreases the amount of structure ascribed to S, nontrivial choices arise in defining such theories. An interesting example is given by taking S to be a semicategory of paths in a directed graph, with operation defined by concatenation. In this case, the operation is only partially defined, which complicates the choice of complexes used to define the cohomology of S.

2. **Algebra cohomology theories, applied to $R[S;\Theta]$, particularly $R[S] := R[S;\mathbf{1}]$.** Familiar algebra cohomology theories include Hochschild, cyclic, and André-Quillen cohomology. If S has "enough structure," then the "obvious choices" of cohomology theories for S and $R[S]$ tend to coincide. This is not true to semicategories in general; for instance, path semicategories provide counterexamples.

3. **Extensions of S.** The prototypical example is extensions of groups. Extensions tend to admit classification by cohomology objects; in particular, $H^2(S)$. This is true for a wide variety of different group-like objects and cohomology theories.

4. **Properties of $R[S;\Theta]$.** Properties of interest include existence of an identity element, commutativity, and associativity. The twisted semicategory algebra $R[S;\Theta]$ generally fails to satisfy such properties even if $R[S]$ satisfies them, unless one imposes special conditions on the twisting map Θ.

5. **Properties of Θ.** Properties of interest include symmetry and the cocycle condition. For example, it is typically true that $R[S;\Theta]$ is associative if and only if Θ is a 2-cocycle. This means that Θ defines a cohomology class in $H^2(S)$. In particular, this means that associative twisted semicategory algebras typically correspond to extensions of the underlying semicategory.