INFINITESIMAL STRUCTURE OF CHOW GROUPS AND ALGEBRAIC K-THEORY

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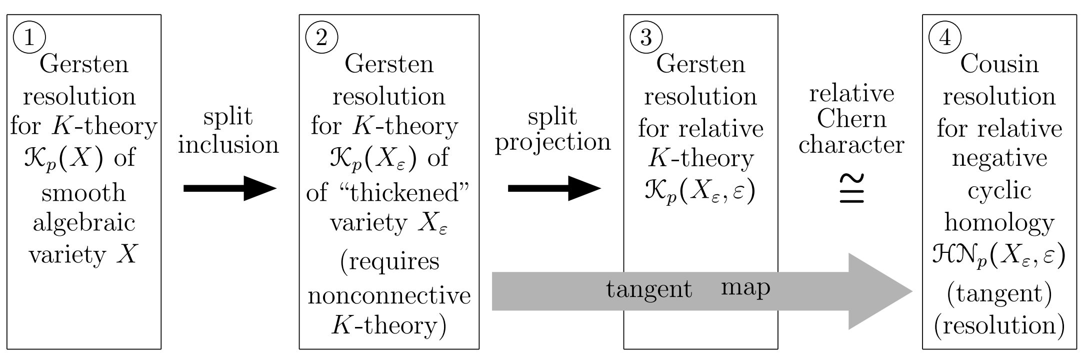
1. Introduction.

The Chow groups $\mathrm{CH}^p(X)$ of codimension-p algebraic cycles modulo rational equivalence, on a projective nonsingular algebraic variety X over a field k containing the rational numbers, remain poorly understood despite intensive study by algebraic geometers over the last half-century. Green and Griffiths [1] have recently introduced an interesting avenue of investigation into the structure of the Chow groups: the study of their tangent groups at the origin $T\mathrm{CH}^p(X)$. This approach is analogous to the study of Lie algebras to elucidate the structure of Lie groups. Here we present a "machine" which extends, generalizes, and formalizes this method. This machine may be represented by the following schematic diagram:



The *Gersten resolution* [6, 3, 2] of the algebraic K-theory sheaf $\mathcal{K}_p(X)$, which appears in the first column, arises from the coniveau filtration of X [19], and may be used to compute the sheaf cohomology groups $H^p(X, \mathcal{K}_p)$, which are isomorphic to the Chow groups $\mathrm{CH}^p(X)$ by Bloch's theorem [5, 3]. Methods of Colliot-Thélène, Hoobler, and Kahn [10], involving *nonconnective K-theory,* permit similar resolution of the K-theory of the "thickened" scheme X_ε defined locally by tensoring with the dual numbers. This produces the second column, whose objects may be viewed as sheaves of infinitesimal deformations, or "arcs," of elements of the corresponding sheaves in the first column, up to a first-order equivalence relation.

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The split surjection of the dual numbers onto the base field induces a splitting of complexes which yields the resolution of relative K-theory appearing in the third column. The *relative Chern character* from relative K-theory to relative negative cyclic homology induces an isomorphism of this resolution with the *Cousin resolution* of the *relative negative cyclic homology* sheaf $\mathcal{HN}_p(X_\varepsilon, X)$, which also arises from the coniveau filtration. This resolution, called the *tangent resolution,* appears in the fourth column. The cohomology groups $H^p\big(X, \mathcal{HN}_p(X_\varepsilon, X)\big)$ of the relative negative cyclic homology sheaf are the *tangent groups at the origin* $T\mathrm{CH}^p(X)$ of the Chow groups $\mathrm{CH}^p(X)$, and are the desired objects of study. The *tangent map,* which is the composition of the split projection and the relative Chern character, assigns a *tangent element* to each equivalence class of arcs in the second column, and induces analogous maps on cohomology. In particular, it assigns a tangent element in $T\mathrm{CH}^p(X)$ to each equivalence class of arcs of in $\mathrm{CH}^p(X)$.

This paper is organized as follows: in section 2, we discuss the definition of the tangent groups $T\mathrm{CH}^p(X)$. This involves choosing an extension of the Chow functor CH^p to a category including certain singular schemes. We choose a K-theoretic extension, motivated by Bloch's theorem. We also review the definition used by Green and Griffiths, which involves only the *Milnor* part of K-theory, and explain why this definition neglects potentially valuable information. In section 3, we describe the case of the tangent group at the origin to the Chow group of zero-cycles on a smooth algebraic surface, studied by Green and Griffiths. Here a rudimentary version of the machine first appears, involving Milnor K-theory and *absolute Kähler differentials,* rather than K-theory and negative cyclic homology. We explain how Green and Griffiths use the Gersten and Cousin resolutions to define the first and fourth columns of the machine.

In sections 4, 5, 6, and 7, we assemble the elements necessary to build the full version of the machine. These are nonconnective K-theory, negative cyclic homology, the coniveau filtration and associated sequences, and the relative Chern character. There are two points of particular importance. First, K-theory and negative cyclic homology may be viewed as *substrata of spectra* inducing *cohomology theories with supports* in the sense of Colliot-Thélène, Hoobler, and Kahn [10], which satisfy suitable *localization* and *projective bundle* formulae, and which therefore admit construction of the necessary resolutions even in the presence of nilpotents. Second, the relative Chern character induces an isomorphism of spectra that respects the *Adams operations* on K-theory and negative cyclic homology. This allows the machine to be decomposed into separate parts for each *Adams weight.* The construction of Green and Griffiths gives only the top-weight part. The other parts may involve interesting new invariants.

2. Definition of the tangent group $T\mathrm{CH}^p(X)$.

The definition of the tangent group at the origin $T\mathrm{CH}^p(X)$ of the Chow group $\mathrm{CH}^p(X)$ is somewhat subtle and involves a choice of *extension of the Chow functor* CH^p. After presenting our definition, we discuss why the problem is not straightforward, compare our definition to the definition of Green and Griffiths, and mention the desirability of a universal definition.

2.1. **Our definition of $T\mathrm{CH}^p$.** Motivated by Bloch's theorem

$$\mathrm{CH}^p(X) \cong H^p(X, \mathcal{K}_p), \tag{2.1.1}$$

we make the definition

$$T\mathrm{CH}^p(X) \coloneqq TH^p(X, \mathcal{K}_p) = H^p(X, T\mathcal{K}_p), \tag{2.1.2}$$

where $\mathcal{K}_p = \mathcal{K}_p(X)$ is the sheaf of Quillen or Thomason K-theory on X. Since X is nonsingular, these two theories coincide for X. By the usual definition of a tangent functor (see, for instance, [8] page 205), we define the tangent sheaf $T\mathcal{K}_p$ to be the kernel

$$T\mathcal{K}_p(X) \coloneqq \mathrm{Ker}\big[\mathcal{K}_p(X_\varepsilon) \to \mathcal{K}_p(X)\big], \tag{2.1.3}$$

where X_ε means $X \times_{Spec(k)} \mathrm{Spec}\ (k[\varepsilon]/\varepsilon^2)$. Moving the tangent operation T inside H^p is justified because H^p is a middle-exact functor.

2.2. **Why not a direct definition in terms of CH^p?** Naïve application of the usual definition of the tangent functor suggests the following definition for $T\mathrm{CH}^p$:

$$T\mathrm{CH}^p(X) \overset{?}{\coloneqq} \mathrm{Ker}\big[\mathrm{CH}^p(X_\varepsilon) \to \mathrm{CH}^p(X)\big].$$

The problem with this definition is the *ambiguous meaning* of $\mathrm{CH}^p(X_\varepsilon)$, since X_ε is singular. For this definition to make sense, we must extend the definition of the Chow functor to a category including schemes like X_ε. The obvious choice, motivated again by Bloch's theorem, is to choose the extension

$$\mathrm{CH}^p(X_\varepsilon) \overset{?}{\coloneqq} H^p(X_\varepsilon, \mathcal{K}_p),$$

since the sheaf cohomology groups $H^p(X_\varepsilon, \mathcal{K}_p)$ are well-defined. However, there is a uniqueness problem: there exist *other* functors besides $H^p(-, \mathcal{K}_p)$ extending CH^p and defined on a category including X_ε. An important example is the functor $X \mapsto H^p(X, \mathcal{K}_p^{\mathrm{M}})$, where $\mathcal{K}_p^{\mathrm{M}}$ is the sheaf of Milnor K-groups on X. This is the extension used by Green and Griffiths.

2.3. **Comparison to Green and Griffiths' definition using Milnor K-theory.** Green and Griffiths focus on the case of $\mathrm{CH}^2(X)$, where X is a *smooth projective algebraic surface,* and use the definition

$$TCH^2_{GG}(X) := TH^2(X, \mathcal{K}_2^{\mathrm{M}}) = H^2(X, T\mathcal{K}_2^{\mathrm{M}}) = H^2(X, \Omega^1_{X/\mathbb{Q}}), \tag{2.3.1}$$

where $\Omega^1_{X/\mathbb{Q}}$ is the sheaf of absolute Kähler differentials on X. In this case, there is no difference between K-theory and Milnor K-theory, since $K_2 = K_2^{\mathrm{M}}$ for regular local rings. However, in the general case, the functors $H^p(-, \mathcal{K}_p^{\mathrm{M}})$ contain *less information* than the functors $H^p(-, \mathcal{K}_p)$, because Milnor K-theory is only the highest-weight Adams eigenspace of K-theory. For example,

$$T\mathcal{K}_3^{\mathrm{M}}(X) \cong \Omega^2_{X/\mathbb{Q}} \quad \text{but} \quad T\mathcal{K}_3(X) \cong \Omega^2_{X/\mathbb{Q}} \oplus \mathcal{O}_X.$$

The additional factor may lead to interesting invariants inaccessible to the approach of Green and Griffiths.

2.4. **Desirability of a universal extension of** CH^p**.** In general, there are injections

$$T\mathrm{CH}^p_{GG}(X) \to T\mathrm{CH}^p(X)$$

for every X and p, so $T\mathrm{CH}^p$ as we have defined it captures more information than $T\mathrm{CH}^p_{GG}$. However, we have not proven that our definition gives the *best possible* extension of CH^p in this regard. Ideally, one would like to have *universal extension functors* of CH^p admitting injective maps from every other extension. It is not clear how to define such functors; in particular, there is no a priori reason why they should come from sheaf cohomology.

3. Green and Griffiths' method for studying $T\mathrm{CH}^2$ of a surface.

3.1. **The tangent sequence for $\mathcal{K}_2^{\mathrm{M}}(X)$.** The canonical split surjection $k[\varepsilon]/\varepsilon^2 \to k$ from the dual numbers onto the base field induces the following split exact sequence, called the tangent sequence:

$$0 \to \mathcal{K}_2^{\mathrm{M}}(X) \overset{incl}{\to} \mathcal{K}_2^{\mathrm{M}}(X_\varepsilon) \overset{proj}{\to} T\mathcal{K}_2^{\mathrm{M}}(X) \to 0. \tag{3.1.1}$$

By Van der Kallen's theorem [12], $T\mathcal{K}_2^{\mathrm{M}}(X) \cong \Omega^1_{X/\mathbb{Q}}$. This isomorphism may be viewed as a special case of the relative Chern character. Thus, the tangent sequence may be rewritten:

$$0 \to \mathcal{K}_2^{\mathrm{M}}(X) \overset{incl}{\to} \mathcal{K}_2^{\mathrm{M}}(X_\varepsilon) \overset{T}{\to} \Omega^1_{X/\mathbb{Q}} \to 0. \tag{3.1.2}$$

The map T is called the *tangent map.* It is the composition of projection and relative Chern character. This version of the tangent map, expressed at the sheaf level, induces a corresponding tangent map at the level of complexes, as described in the next section.

The tangent map is simple enough in this case to describe explicitly. Let

$$\{f_0 + f_1\varepsilon, g_0 + g_1\varepsilon\} \tag{3.1.3}$$

be a *Steinberg symbol* representing an element of $\mathcal{K}_2^{\mathrm{M}}(X_\varepsilon)(U)$ for some open set $U \subset X$. Projection to the relative part $\mathcal{K}_2^{\mathrm{M}}(X_\varepsilon,\varepsilon)$ "peels off the constant symbol $\{f_0, g_0\}$," leaving the product

$$\Big\{f_0, 1+\frac{g_1}{g_0}\varepsilon\Big\}\Big\{1+\frac{f_1}{f_0}\varepsilon, g_0\Big\}\Big\{1+\frac{f_1}{f_0}\varepsilon, 1+\frac{g_1}{g_0}\varepsilon\Big\}. \tag{3.1.4}$$

Applying the relative Chern character then yields the differential

$$\frac{g_1}{g_0}\frac{df_0}{f_0} - \frac{f_1}{f_0}\frac{dg_0}{g_0} \quad \in \quad \Omega^1_{X/\mathbb{Q}}(U). \tag{3.1.5}$$

Passage from 3.1.4 to 3.1.5 is a special case of Maazen and Stienstra's logarithm formula [13]:

$$\{a,b\} \mapsto \frac{1}{a}\log(b)da, \tag{3.1.6}$$

which remains valid when the algebra of dual numbers is replaced with an arbitrary *local artinian k-algebra*. Each symbol appearing in the product in equation 3.1.4 is an "infinitesimal arc through the origin in $\mathcal{K}_2^{\mathrm{M}}(X_\varepsilon)(U)$," in the sense that replacing ε with zero yields trivial symbols. The terms $(g_1/g_0)(df_0/f_0)$ and $-(f_1/f_0)(dg_0/g_0)$ come from the first two factors, respectively, while the final factor is trivial by the computations appearing on page 79 of Green and Griffiths [1]. The differential in equation 3.1.5 should really be multiplied by ε, but the factor of ε may be dropped under the identification $\Omega^1_{X/\mathbb{Q}}(U)\otimes_k(\varepsilon) \leftrightarrow \Omega^1_{X/\mathbb{Q}}(U)$.

3.2. **Gersten and Cousin resolutions; first glimpse of the machine.** Using the Gersten resultion of $\mathcal{K}_2^{\mathrm{M}}(X)$ and the Cousin resolution of $\Omega^1_{X/\mathbb{Q}}$, we obtain from the tangent sequence the following "horseshoe diagram" with exact row and columns:

$$\begin{array}{ccccc}
\mathcal{K}_2^{\mathrm{M}}(X) & \xrightarrow{incl} & \mathcal{K}_2^{\mathrm{M}}(X_\varepsilon) & \xrightarrow{T} & \Omega^1_{X/\mathbb{Q}} \\
\downarrow{\scriptstyle d_0} & & & & \downarrow{\scriptstyle Td_0} \\
K_2^{\mathrm{M}}(k(\eta)) & & & & H^0_\eta(\Omega^1_{X/\mathbb{Q}}) \\
\downarrow{\scriptstyle d_1} & & & & \downarrow{\scriptstyle Td_1} \\
\bigoplus_{x\in X^{(1)}} K_1^{\mathrm{M}}(k(x)) & & & & \bigoplus_{x\in X^{(1)}} H^1_x(\Omega^1_{X/\mathbb{Q}}) \\
\downarrow{\scriptstyle d_2} & & & & \downarrow{\scriptstyle Td_2} \\
\bigoplus_{x\in X^{(2)}} K_0^{\mathrm{M}}(k(x)) & & & & \bigoplus_{x\in X^{(2)}} H^2_x(\Omega^1_{X/\mathbb{Q}})
\end{array} \tag{3.2.1}$$

Here, η is the *generic point* of X. For each $q \geq 0$, $K_{2-q}^{\mathrm{M}}\big(k(x)\big)$ denotes the *skyscraper sheaf* $(i_x)_* K_{2-q}^{\mathrm{M}}\big(k(x)\big)$ at a codimension-q point x of X corresponding to the group $K_{2-q}^{\mathrm{M}}\big(k(x)\big)$, where $k(x)$ is the *residue field* of x. Similarly, $H_x^{2-q}(\Omega^1_{X/\mathbb{Q}})$ denotes the skyscraper sheaf $(i_x)_* H_x^{2-q}(\Omega^1_{X/\mathbb{Q}})$ at x corresponding to the *local cohomology group* $H_x^{2-q}(\Omega^1_{X/\mathbb{Q}})$. The term $K_2^{\mathrm{M}}\big(k(\eta)\big)$ is the *constant sheaf* on X corresponding to the group $K_2^{\mathrm{M}}(k(X))$, where $k(X)$ is the *function field* of X, and the term $H_\eta^0(\Omega^1_{X/\mathbb{Q}})$ is the constant sheaf on X corresponding to the *module* $\Omega^1_{k(X)/\mathbb{Q}}$.

The left column is the first column in our schematic diagram of the machine, and the right column is the fourth column. The sheaf $\mathcal{K}_2^{\mathrm{M}}\big(X_\varepsilon\big)$ is the initial object in the second column, which has not yet been constructed. The third column does not appear in this version of the machine because the tangent map combines projection and relative Chern character, and Green and Griffiths do not separate these two maps in their treatment.

The bottom term $\bigoplus_{x \in X^{(2)}} K_0^{\mathrm{M}}\big(k(x)\big)$ in the right column is the *sheaf of codimension-*2 *cycles* on X. The image of the map induced by d_2 via the global section functor is the group of rational equivalences, so the corresponding quotient is the Chow group by the definition of sheaf cohomology and Bloch's theorem:

$$\mathrm{CH}^2(X) = \frac{\Gamma \bigoplus_{x \in X^{(2)}} K_0^{\mathrm{M}}\big(k(x)\big)}{\mathrm{Im}\ \Gamma d_2}. \tag{3.2.2}$$

Green and Griffiths' definition 2.3.1 of the tangent group, repeated below,

$$T\mathrm{CH}^2_{GG}(X) \coloneqq TH^2(X, \mathcal{K}_2^{\mathrm{M}}) = H^2(X, T\mathcal{K}_2^{\mathrm{M}}) = H^2(X, \Omega^1_{X/\mathbb{Q}}),$$

together with the fact that the right column is a flasque resolution of $\Omega^1_{X/\mathbb{Q}}$, gives an analogous formula for the tangent group $T\mathrm{CH}^2(X)$:

$$T\mathrm{CH}^2_{GG}(X) \coloneqq \frac{\Gamma \bigoplus_{x \in X^{(2)}} H_x^2(\Omega^1_{X/\mathbb{Q}})}{\mathrm{Im}\ \Gamma T d_2}. \tag{3.2.3}$$

3.3. **Arc objects.** Green and Griffiths furnish a *geometric interpretation* of $T\mathrm{CH}^2_{GG}(X)$ in terms of what they call "arcs of cycles." Such "arcs" are described as the *vanishing loci of pairs* $\big(f(t), g(t)\big)$ of functions on X that vary with t. Green and Griffiths view the family of such arcs as an object

$$\text{“Arcs}\Big(\bigoplus_{x \in X^{(2)}} K_0^{\mathrm{M}}\big(k(x)\big)\Big),\text{”}$$

which is never explicitly defined. Computing the "tangent" of an arc of cycles involves imposing a relation of "first-order equivalence" $\sim_1$ on such "arcs," which is also treated informally. The result is an element of the final object

$$\bigoplus_{x\in X^{(2)}} H^2_x(\Omega^1_{X/\mathbb{Q}}),$$

in the Cousin resolution of $\Omega^1_{X/\mathbb{Q}}$. This object is identified as the tangent sheaf of the sheaf of codimension-2 cycles on X.

In addition to "arcs of cycles," Green and Griffths invoke informal "arc objects" corresponding to the remaining terms of the Gersten resolution of $\mathcal{K}^{\mathrm{M}}_2(X)$. They use these "arc objects" to "fill in the horseshoe" and obtain the following diagram, where the maps involving the middle column are "defined" partly by analogy, partly by concrete geometric arguments, and partly by the requirement that the resulting diagram be commutative.

(3.3.1)
$$\begin{array}{ccccc}
\mathcal{K}^{\mathrm{M}}_2(X) & \xrightarrow{\ incl\ } & \mathcal{K}^{\mathrm{M}}_2(X_\varepsilon) & \xrightarrow{\ T\ } & \Omega^1_{X/\mathbb{Q}} \\
\downarrow{\scriptstyle d_0} & & \downarrow & & \downarrow{\scriptstyle Td_0} \\
K^{\mathrm{M}}_2\big(k(\eta)\big) & \longrightarrow & \dfrac{\mathrm{Arcs}\big(\mathcal{K}^{\mathrm{M}}_2\big(k(X)\big)\big)}{\sim_1} & \longrightarrow & H^0_\eta(\Omega^1_{X/\mathbb{Q}}) \\
\downarrow{\scriptstyle d_1} & & \downarrow & & \downarrow{\scriptstyle Td_1} \\
\bigoplus_{x\in X^{(1)}} K^{\mathrm{M}}_1\big(k(x)\big) & \longrightarrow & \dfrac{\mathrm{Arcs}\Big(\bigoplus_{x\in X^{(1)}} K^{\mathrm{M}}_1\big(k(x)\big)\Big)}{\sim_1} & \longrightarrow & \bigoplus_{x\in X^{(1)}} H^1_x(\Omega^1_{X/\mathbb{Q}}) \\
\downarrow{\scriptstyle d_2} & & \downarrow & & \downarrow{\scriptstyle Td_2} \\
\bigoplus_{x\in X^{(2)}} K^{\mathrm{M}}_0\big(k(x)\big) & \longrightarrow & \dfrac{\mathrm{Arcs}\Big(\bigoplus_{x\in X^{(2)}} K^{\mathrm{M}}_0\big(k(x)\big)\Big)}{\sim_1} & \longrightarrow & \bigoplus_{x\in X^{(2)}} H^2_x(\Omega^1_{X/\mathbb{Q}})
\end{array}$$

3.4. **Significance of the concrete viewpoint.** The nonuniqueness of the functor $X \mapsto H^p(X, \mathcal{K}_p)$ as an extension of the Chow functor CH^p makes the concrete viewpoint adopted by Green and Griffiths particularly important, since it shows directly that their method has geometric content. The arc object

$$\mathrm{Arcs}\Big(\bigoplus_{x\in X^{(2)}} K^{\mathrm{M}}_0\big(k(x)\big)\Big),$$

though ill-defined, has the advantage that its elements *manifestly represent arcs of cycles,* in the most elementary sense of one-parameter families. Any satisfactory definition of the tangent space should include tangents to such arcs.

4. Replacing Milnor K-theory with nonconnective K-theory.

To properly define the arc objects modulo first order equivalence introduced by Green and Griffiths, we use the *nonconnective K-theory* of Bass and Thomason. Nonconnective K-theory satisfies certain conditions, such as the hypotheses of *Thomason's localization theorem,* that permit the construction of Gersten resolutions in more general settings than Quillen K-theory. For the first column of the machine, which involves the nonsingular variety X, Quillen K-theory is sufficient; here Quillen's *devissage* theorem may be used to express the local terms appearing in the Gersten resolution as Quillen K-groups of residue fields. For the second column, which involves the singular scheme X_ε, *devissage* does not apply, and the local terms must be expressed as nonconnective K-groups.

4.1. **Definition and properties.** For a pair (X, Z), where X is a quasi-compact, quasi-separated scheme of finite Krull dimension, and Z is a closed subspace of X such that $X - Z$ is also quasi-compact, Thomason's *nonconnective K-theory spectrum* $\mathbf{K}(X \text{ on } Z)$ is defined ([4] definition 6.4 page 360) as the homotopy colimit of a diagram

$$F^0 \to F^{-1} \to F^{-2} \to ..., \tag{4.1.1}$$

where F^0 is the *connective K-theory spectrum,* which in turn is defined ([4] definition 3.1 page 313) to be the K-theory spectrum of the *complicial biWaldhausen category of those perfect complexes on X which are acyclic on $X - Y$*. Given a point $x \in X$, the spectrum $\mathbf{K}(X \text{ on } x)$ is defined to be direct limit $\varinjlim_{U \ni x} \mathbf{K}(U \text{ on } \bar{x} \cap U)$ of spectra over open sets U containing x.

A crucial technical property of nonconnective K-theory in the present context is that it satisfies Thomason's localization theorem ([4] Theorem 7.4 page 365), which states that there is a *homotopy fiber sequence*

$$\mathbf{K}(X \text{ on } Z) \to \mathbf{K}(X) \to \mathbf{K}(X - Z). \tag{4.1.2}$$

This result, together with a few other technical details, implies that nonconnective K-theory is a substratum of spectra satisfying the *étale Mayer-Vietoris* and *projective bundle* properties in the sense of Colliot-Thélène, Hoobler, and Kahn [10]. This guarantees the existence of the desired Gersten resolutions.

4.2. **First three columns of the machine.** Using nonconnective K-theory, together with the results of [10], yields the first two columns of our machine. Because of the splitting, the relative objects in the third column are just the cokernels of the corresponding terms in the first two columns. This produces the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccc}
\mathcal{K}_p(X) & \longrightarrow & \mathcal{K}_p(X_\varepsilon) & \longrightarrow & \mathcal{K}_p(X_\varepsilon, X) \\
\downarrow & & \downarrow & & \downarrow \\
K_p(X \text{ on } \eta) & \longrightarrow & K_p(X_\varepsilon \text{ on } \eta) & \longrightarrow & K_p(X_\varepsilon, X \text{ on } \eta) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x\in X^{(1)}} K_{p-1}(X \text{ on } x) & \longrightarrow & \bigoplus_{x\in X^{(1)}} K_{p-1}(X_\varepsilon \text{ on } x) & \longrightarrow & \bigoplus_{x\in X^{(1)}} K_{p-1}(X_\varepsilon, X \text{ on } x) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x\in X^{(p-1)}} K_1(X \text{ on } x) & \longrightarrow & \bigoplus_{x\in X^{(p-1)}} K_1(X_\varepsilon \text{ on } x) & \longrightarrow & \bigoplus_{x\in X^{(p-1)}} K_1(X_\varepsilon, X \text{ on } x) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x\in X^{(p)}} K_0(X \text{ on } x) & \longrightarrow & \bigoplus_{x\in X^{(p)}} K_0(X_\varepsilon \text{ on } x) & \longrightarrow & \bigoplus_{x\in X^{(p)}} K_0(X_\varepsilon, X \text{ on } x)
\end{array}
\tag{4.2.1}
$$

4.3. **Re-definition of the arc objects modulo first-order equivalence.** This diagram motivates the following formal definition of the arc objects modulo first-order equivalence introduced by Green and Griffiths:

$$
\frac{\operatorname{Arcs}\Big(\bigoplus_{x\in X^{(p-q)}} K_q^{\mathrm{M}}\big(X \text{ on } x\big)\Big)}{\sim_1} := \bigoplus_{x\in X^{(p-q)}} K_q(X_\varepsilon \text{ on } x). \tag{4.3.1}
$$

This definition allows us to work with well-defined objects and treats the first and second columns of the machine in a unified way. It also generalizes the work of Green-Griffiths in the sense that every tangent computed by Green-Griffiths is also a tangent under the new definition.

5. Replacing Kähler differentials with negative cyclic homology

The fourth and final column of our machine is given by the Cousin resolution of the relative negative cyclic homology sheaf $\mathcal{HN}_p(X_\varepsilon, X)$ on X. By *Hesselholt's theorem* [11], this sheaf may be expressed in this context in terms of absolute Kähler differentials:

$$
\mathcal{HN}_p(X_\varepsilon, X) \cong \Omega^{p-1}_{X/\mathbb{Q}} \oplus \Omega^{p-3}_{X/\mathbb{Q}} \oplus \ldots \tag{5.0.1}
$$

However, we retain the more general expression because of its conceptual advantages. In particular, negative cyclic homology is the proper receptacle for the relative Chern character, and the general expression also extends to the case where the algebra of dual numbers $k[\varepsilon]/\varepsilon^2$ is replaced by an arbitrary local artinian k-algebra.

The idea that cyclic homology is a good place to look for infinitesimal invariants of Chow groups is partly motivated by the fact that the cyclic homology of an algebra corresponds to its Lie algebra of matrices in approximately the same way in which its algebraic K-theory corresponds to its general linear group. For example, see [9], 10.2.19 and 11.2.12, and [18], theorem A.15.

5.1. **Definition and properties.** The negative cyclic homology of a scheme X may be defined [17] in terms of the *Zariski hypercohomology* of a complex $\mathcal{HN}^*(X)$ of sheaves on X. Using this formulation, the complex $\mathcal{HN}^*(X)$ is the *product total complex* of Connes' sheafified (B,b)-*bicomplex* (see [9] section 2.1), and the Zariski hypercohomology is defined as

$$\mathbb{H}^p_{zar}\big(\mathcal{HN}^*(X)\big) \coloneqq H^p\big(\Gamma(\mathrm{Tot}\ \mathcal{I}^{**})\big), \tag{5.1.1}$$

where $\mathcal{I}^{**}$ is a *Cartan-Eilenberg resolution* of $\mathcal{HN}^*(X)$.

For some purposes, particularly when functorial properties are crucial, it is convenient to use the equivalent but more general definition of negative cyclic homology given by Keller in [15] and used by Cortiñas et al. in [16]. This definition uses Keller's general machinery of *localization pairs* [14, 15] to obtain a *mixed complex* $C(X)$; negative cyclic homology is defined in terms of this mixed complex as in [16] section 2. Note that [16] also gives a good description (example 2.7 page 8) of the relevant localization pair, which consists of the category of perfect complexes on X and its *acyclic subcategory,* modified slightly to deal with cardinality issues. Example 2.8 in [16] gives a version with supports, which, after taking limits, defines the groups $\mathrm{HN}_p(X \text{ on } x)$ appearing in the Cousin resolution.

To be precise, the functor **HN**, defined in [16], is a presheaf of complexes constructed as follows:

- Let $(\mathbf{Ch}_{parf}(X), \mathcal{A}c)$ be the localization pair formed by the category of perfect complexes which are acyclic on X, and its full subcategory of acyclic complexes, modified to deal with cardinality issues as mentioned above.
- Keller's machinery of localization pairs ([15], [14]) then produces a mixed complex $\mathbf{C}(X) = (C, b, B)(X)$ over k.
- $\mathbf{HN}(X)$ is defined to be the following total complex (in the sense of [16] page 6; note that it is *neither* the sum nor product total complex)

$$\mathrm{Tot}(\ldots \to 0 \to C \xrightarrow{B} C[-1] \xrightarrow{B} C[-2] \xrightarrow{B} \ldots)$$

The *Eilenberg-MacLane functor* from complexes to spectra [21] converts **HN** to a presheaf of spectra on the Zariski site. From this viewpoint, negative cyclic homology is a *substratum of spectra* satisfying the étale Mayer-Vietoris and projective bundle properties, and may thus be treated on the same footing as nonconnective K-theory. For the étale Mayer-Vietoris property, see [16] theorem 2.9 and the discussion in section 3. For the projective bundle formula, see remark 2.11 in the same paper.

6. Coniveau filtration and related resolutions.

6.1. **Coniveau generalities.** The resolutions appearing in our machine all arise from the *coniveau filtration* of X, which is a special case of *filtration of a topological space by sheaves of families of supports,* as described by Hartshorne [19]. "Coniveau" is a French term meaning "codimension" in this context. The coniveau filtration produces *coniveau spectral sequences* for cohomology theories with supports. For algebraic K-theory, the coniveau spectral sequence is

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} K_{-p-q}(X \text{ on } x) \Rightarrow K_{-p-q}(X). \tag{6.1.1}$$

The $-q$th line of the coniveau spectral sequence is the qth Gersten complex of X. Quillen and Gersten were the first to study this sequence for higher K-theory, working in the connective setting and using Quillen's Q-construction. Quillen's *devissage* theorem applies in this context and permits expression of the local terms $K_{-p-q}(X \text{ on } x)$ as K-groups of residue fields $K_{-p-q}(k(x))$. Quillen [3] used this machinery to extend Bloch's theorem 2.1.1, which Bloch had already proven for CH^2 in certain cases [5]. Gersten [6] conjectured that the Gersten resolution is exact for the spectrum of a regular local ring. This explains the "Bloch-Gersten-Quillen" nomenclature.

6.2. **Specific background for our construction.** Colliot-Thélène, Hoobler, and Khan [10], and Balmer [2], later generalized the coniveau spectral sequence and Gersten resolutions to the nonconnective setting. In [10], these specific constructions appear obliquely (see example 7.4(6) page 43 and Theorem 8.1.1 page 46) as examples of a more general construction of coniveau spectral sequences and Cousin complexes for cohomology theories with supports. In [2] they arise from a specifically K-theoretic construction.

Colliot-Thélène, Hoobler, and Khan prove that a *substratum of spectra* produces a coniveau spectral sequence and Gersten complexes of the above form (see [10] Remarks 5.1.3). In this general setting, the Gersten complexes for K-theory are viewed as Cousin complexes in the sense of [19]. If, moreover, the substratum satisfies the étale Mayer-Vietoris and projective bundle properties, then the sheafified Cousin complexes (in this case, Gersten complexes) are flasque resolutions (see [10] Corollary 5.1.11). In particular, the sheafified Gersten complexes for algebraic K-theory over a projective nonsingular algebraic variety are flasque resolutions. Further, these complexes are *universally exact* in the sense of [10] Definition 6.1.1. In particular this means that these complexes remain flasque resolutions after multiplying by a fixed variety, not necessary smooth (see [10] 8.1). This implies that the Gersten complex for the thickened variety X_ε given by multiplying X by the spectrum of the dual numbers is a flasque resolution.

Negative cyclic homology may be treated in an analogous way. The coniveau filtration produces a coniveau spectral sequence

$$E_1^{p,q} = \coprod_{x\in X^{(p)}} \mathrm{HN}_{-p-q}(X \text{ on } x) \Rightarrow \mathrm{HN}_{-p-q}(X), \tag{6.2.1}$$

and Cousin complexes are defined by fixing q. The Cousin complexes are flasque resolutions and are universally exact.

7. Relative algebraic Chern character and the tangent map

7.1. **Background.** The generalized Chern character is a natural transformation of functors from algebraic K-theory to negative cyclic homology. The relative version of the Chern character, which is closely related to Goodwillie's isomorphism [24], induces an isomorphism of complexes between the Gersten resolution of relative K-theory and the Cousin resolution of relative negative cyclic homology. Loday [9] gives a good treatment of the Chern character in the modern algebraic setting, and compares this to the more familiar classical version. Cortiãs and Weibel [22] give an excellent discussion of relative Chern characters for nilpotent ideals. The exposition in this section closely follows these two sources.

Classically, the Chern character computes an invariant of K-theory, topological or algebraic, with values in de Rham cohomology. An algebraic version may be defined by replacing vector bundles with finitely-generated projective modules, as described in Loday [9], section 8.1. In modern settings, cyclic homology, or its periodic or negative variant, is often substituted for de Rham cohomology. This replacement is essential in the case of noncommutative algebras, where the classical definitions of differential forms, and hence de Rham cohomology, do not apply. In the commutative case, the Chern character to cyclic homology lifts the classical Chern character, so nothing is lost by changing the target to cyclic homology. As we noted above, the relationships among cyclic homology, Lie algebras of matrices, algebraic K-theory, and general linear groups suggest that the Chern character to *negative* cyclic homology is the appropriate version in this context.

7.2. **Modern Construction.** Loday [9] describes how the classical Chern character may be progressively generalized, first to a map from $K_0(R)$ to cyclic homology, then to a map from $K_n(R)$ to negative cyclic homology, and finally to a relative version

$$\mathrm{ch}_n : K_n(R,I) \to \mathrm{HN}_n(R,I). \tag{7.2.1}$$

Loday uses the *relative Volodin construction* to describe ch_n^-. In particular, there exists a map of complexes

$$C_\bullet\big(X(R,I)\big) \to \mathrm{Ker}\big[\mathrm{ToT\ CN}(R) \to \mathrm{ToT\ CN}(R,I)\big], \tag{7.2.2}$$

where $X_(R,I)$ is an appropriate *relative Volodin-type space*, $C_\bullet$ is the *Eilenberg-MacLane complex*, CN_R is the *negative cyclic bicomplex* of R, $\mathrm{CN}_(R,I)$ is the relative negative cyclic bicomplex, defined as the kernel of the map $\mathrm{CN}(R) \to \mathrm{CN}(R/I)$

(2.1.15), and ToT is the total complex whose degree-n term is $\prod_{p+q=n} \mathrm{CN}_{p,q}$. Taking homology gives a map

$$H_\bullet\big(X(R,I)\big) \to \mathrm{HN}(R,I). \tag{7.2.3}$$

Composing on the left with the Hurewicz map

$$K_n(R,I) = \pi_n\big(X(R,I)^+\big) \to H_n\big(X(R,I)\big), \tag{7.2.4}$$

yields the desired relative Chern character.

If R is a $\mathbb{Q}$-algebra and I is a two-sided nilpotent ideal in R, then the composite map

$$K_{n,R,I} \xrightarrow{\rho} \mathrm{HC}_{n-1,R,I} \xrightarrow{B} \mathrm{HN}_{n,R,I},$$

is the relative Chern character, where ρ is Goodwillie's isomorphism, and B is Connes' boundary map.

7.3. **Ladder Diagram.** The absolute and relative Chern characters fit into the following commutative "ladder diagram" with exact rows.

$$\begin{array}{ccccccccccc}
\longrightarrow & K_{n+1}(R) & \longrightarrow & K_{n+1}(R/I) & \longrightarrow & K_n(R,I) & \longrightarrow & K_n(R) & \longrightarrow & K_n(R/I) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \mathrm{ch}_{n+1,R}} & & \downarrow{\scriptstyle \mathrm{ch}_{n+1,R/I}} & & \downarrow{\scriptstyle \mathrm{ch}_{n,R,I}} & & \downarrow{\scriptstyle \mathrm{ch}_{n,R}} & & \downarrow{\scriptstyle \mathrm{ch}_{n,R/I}} & \\
\cdots \longrightarrow & \mathrm{HN}_{n+1}(R) & \to & \mathrm{HN}_{n+1}(R/I) & \to & \mathrm{HN}_n(R,I) & \longrightarrow & \mathrm{HN}_n(R) & \longrightarrow & \mathrm{HN}_n(R/I) & \longrightarrow \cdots
\end{array}$$

Loday [9], Proposition 11.4.8, gives a proof of this result.

7.4. **Examples.** If S is a smooth algebra over a commutative ring k containing $\mathbb{Q}$, then

$$\mathrm{HN}_{2,S} \cong Z^2(S) \times \prod_{i>0} H_{\mathrm{dR}}^{2i+2}(S),$$

where $Z^2(S)$ is the kernel of the differential map $d : \Omega^2_{S/k} \to \Omega^3_{S/k}$, and H_{dR} is de Rham cohomology. The quotient of $Z^2(S)$ by exact forms is by definition the second de Rham cohomology group $H^2_{\mathrm{dR}}(S)$, so $\mathrm{HN}_2(S)$ projects into $H^2_{\mathrm{dR}}(S)$. The composite map

$$K_2^{\mathrm{M}}(S) \to K_2(S) \xrightarrow{\mathrm{ch}_2} \mathrm{HN}_2(S) \to H^2_{\mathrm{dR}}(S),$$

is the canonical $d\log$ map sending the Steinberg symbol $\{r, r'\}$ to $(dr/r) \wedge (dr'/r)$ in $H^2_{\mathrm{dR}}(S)$. See Loday [9], page 275, for details.

Now suppose that S is a local ring at a point on a smooth complex algebraic variety, and let $R = S_\varepsilon = S[\varepsilon]/\varepsilon^2$ be the ring of dual numbers over S. The above ladder diagram splits into short exact ladder diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_n\big(S_\varepsilon,\varepsilon\big) & \longrightarrow & K_n(S_\varepsilon) & \longrightarrow & K_n(S) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{ch}_{n,S_\varepsilon,(\varepsilon)}} & & \Big\downarrow{\scriptstyle \mathrm{ch}_{n,S_\varepsilon}} & & \Big\downarrow{\scriptstyle \mathrm{ch}_{n,S}} & & \\
0 & \longrightarrow & \mathrm{HN}_n\big(S_\varepsilon,\varepsilon\big) & \longrightarrow & \mathrm{HN}_n(S_\varepsilon) & \longrightarrow & \mathrm{HN}_n(S) & \longrightarrow & 0
\end{array}$$

For $n = 0$, the relative groups vanish, and the diagram is not very interesting. For $n = 1$, the diagram reduces to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 1+\varepsilon S & \xrightarrow{\text{incl}} & S_\varepsilon^* & \xrightarrow{\varepsilon\mapsto 0} & S^* & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & S & \xrightarrow{d} & Z^1(S_\varepsilon)\times H_{\mathrm{dR}} & \xrightarrow{\varepsilon\mapsto 0} & Z^1(S)\times H_{\mathrm{dR}} & \longrightarrow & 0
\end{array}$$

The first vertical map in the diagram for $n = 1$ is the map sending $1+\varepsilon s$ to s. The proper way to view this map is as a logarithmic map

$$1+\varepsilon s \mapsto \log(1+\varepsilon s) = \varepsilon s - \frac{1}{2}\varepsilon^2 s^2 + \ldots, \tag{7.4.1}$$

where only the first term survives since $\varepsilon = 0$. The resulting element εs may be viewed as an element of S via the identification $S\otimes_k(\varepsilon)\leftrightarrow S$. The remaining vertical maps are version of the $d$log map.

For $n = 2$, the diagram reduces to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_2^{\mathrm{M}}(S_\varepsilon,\varepsilon) & \xrightarrow{\text{incl}} & K_2^{\mathrm{M}}(S_\varepsilon) & \xrightarrow{\varepsilon\mapsto 0} & K_2^{\mathrm{M}}(S) & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & \Omega^1_{S/k} & \xrightarrow{d} & Z^2(S_\varepsilon)\times H_{\mathrm{dR}} & \xrightarrow{\varepsilon\mapsto 0} & Z^2(S)\times H_{\mathrm{dR}} & \longrightarrow & 0
\end{array}$$

The first vertical map in the diagram for $n = 2$ may be expressed in terms of Maazen and Stienstra's logarithm formula, as described in section 3.1 above. The remaining vertical maps are version of the $d$log map.

Due to the functorial properties of the coniveau spectral sequence, the relative Chern character induces morphisms of complexes between the Gersten resolution of relative algebraic K-theory $\mathcal{K}_p(X_\varepsilon,\varepsilon)$ and the Cousin resolution of negative cyclic homology $\mathcal{HN}_p(X_\varepsilon,\varepsilon)$. The composition of the projection map from the Gersten resolution of the augmented K-theory sheaf $\mathcal{K}_p(X_\varepsilon)$ to the Gersten resolution of the relative sheaf $\mathcal{K}_p(X_\varepsilon,\varepsilon)$ and the morphism of complexes induced by the Chern character is the generalized tangent map at the level of complexes.

8. Final Form of the Machine.

The final form of the machine is given by the following commutative diagram:

$$
(8.0.1)\qquad
\begin{array}{ccccccc}
\mathcal{K}_p(X) & \xrightarrow{\text{incl}} & \mathcal{K}_p(X_\varepsilon) & \xrightarrow{\text{proj}} & \mathcal{K}_p(X_\varepsilon,\varepsilon) & \xrightarrow[\sim]{\text{ch}} & \mathcal{HN}_p(X_\varepsilon,\varepsilon) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_p(X \text{ on } \eta) & \xrightarrow{\text{incl}} & K_p(X_\varepsilon \text{ on } \eta) & \xrightarrow{\text{proj}} & K_p(X_\varepsilon,\varepsilon \text{ on } \eta) & \xrightarrow[\sim]{\text{ch}} & \mathrm{HN}_p(X_\varepsilon,\varepsilon \text{ on } \eta) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x\in X^{(1)}} K_{p-1}(X \text{ on } x) & \xrightarrow{\text{incl}} & \bigoplus_{x\in X^{(1)}} K_{p-1}(X_\varepsilon \text{ on } x) & \xrightarrow{\text{proj}} & \bigoplus_{x\in X^{(1)}} K_{p-1}(X_\varepsilon,\varepsilon \text{ on } x) & \xrightarrow[\sim]{\text{ch}} & \bigoplus_{x\in X^{(1)}} \mathrm{HN}_{p-1}(X_\varepsilon,\varepsilon \text{ on } x) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x\in X^{(p-1)}} K_1(X \text{ on } x) & \xrightarrow{\text{incl}} & \bigoplus_{x\in X^{(p-1)}} K_1(X_\varepsilon \text{ on } x) & \xrightarrow{\text{proj}} & \bigoplus_{x\in X^{(p-1)}} K_1(X_\varepsilon,\varepsilon \text{ on } x) & \xrightarrow[\sim]{\text{ch}} & \bigoplus_{x\in X^{(p-1)}} \mathrm{HN}_1(X_\varepsilon,\varepsilon \text{ on } x) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x\in X^{(p)}} K_0(X \text{ on } x) & \xrightarrow{\text{incl}} & \bigoplus_{x\in X^{(p)}} K_0(X_\varepsilon \text{ on } x) & \xrightarrow{\text{proj}} & \bigoplus_{x\in X^{(p)}} K_0(X_\varepsilon,\varepsilon \text{ on } x) & \xrightarrow[\sim]{\text{ch}} & \bigoplus_{x\in X^{(p)}} \mathrm{HN}_0(X_\varepsilon,\varepsilon \text{ on } x)
\end{array}
$$

9. Lambda and Adams operations.

9.1. **Lambda and Adams operations under the Chern character.** In the context of generalized cohomology theories, *lambda* and *Adams operations* are *cohomology operations* that may be used to decompose the corresponding cohomology objects. Lambda operations generalize the exterior power operation, and Adams operations are polynomials in the lambda operations. Given a natural transformation between cohomology theories, an important question is whether or not this transformation respects these operations. In the present context, the cohomology theories of interest are K-theory and negative cyclic homology, and the transformation between them is the relative Chern character $ch : \mathcal{K}_p(X_\varepsilon,\varepsilon) \to \mathcal{HN}_p(X_\varepsilon,\varepsilon)$. Cortiñas et al. [18] have recently published a proof that the relative Chern character, and even the absolute Chern character, does indeed preserve the lambda and Adams operations. This allows us to decompose our machine into separate pieces, one for each piece of the corresponding decompositions in cohomology.

The lambda and Adams operations are defined in different ways for K-theory and negative cyclic homology. For K-theory, see [26] II.4 and IV.5 for a treatment in terms of the plus-construction, which suffices for the top row of the machine, and [27] for an explicit treatment of the nonconnective case. For negative cyclic homology, see [9] chapters 4 and 5.

The version we need may be expressed as follows. Let Y be a scheme with structure sheaf $\mathcal{O}_Y$, $\mathcal{J}$ a nilpotent sheaf of ideals on Y, and $\mathbf{K}(\mathcal{O},\mathcal{J})$ the presheaf of spectra

$$\mathbf{K}(\mathcal{O},\mathcal{J}) : U \mapsto \mathbf{K}(\mathcal{O}(U),\mathcal{J}(U)), \tag{9.1.1}$$

where $U \subset Y$ is an open subset. Define $\mathbf{HN}(\mathcal{O},\mathcal{J})$ in an analogous way. Then the relative Chern character

$$ch : \mathbf{K}(\mathcal{O},\mathcal{J}) \to \mathbf{HN}(\mathcal{O},\mathcal{J}), \tag{9.1.2}$$

is an isomomorphism that preserves the lambda operations up to natural homotopy. For the proof, see [18] Appendix B. The analogous result for Adams operations is an easy corollary.

As noted above, Cortiñas et al. [18] have proven that the *absolute* Chern character respects the lambda and Adams operations. Their proof uses the corresponding result for the relative Chern character in the nilpotent case, which is the case we need, and which is actually easier to prove. This case has been known for some time, but its original proof, given by Cathelineau [25], suffers from an error originating in an unpublished preprint of Ogle and Weibel. This error is corrected, and the implications are carefully explained, in [18], particularly the appendices.

9.2. **Weight shifting in the Gersten complex.** We remark on a subtlety regarding the decomposition of the machine into Adams eigenspaces. The abstract forms of the Gersten and Cousin resolutions in the machine, expressed in terms of cohomology groups with supports, respect the Adams operations. However, recall that the Gersten resolution of $\mathcal{K}_p(X)$ may alternatively be expressed, à la Quillen, in terms of residue fields. In this form, the maps in the Gersten resolution of $\mathcal{K}_p(X)$ have weight shifts; these shifts arise from corresponding weight shifts in the isomorphisms between the abstract groups and with supports and the groups expressed in terms of residue fields. These weight shifts are shown in the following diagram:

$$
\begin{array}{ccc}
\mathcal{K}_p(\mathcal{O}_X) & \overset{0}{\cong} & \mathcal{K}_p(\mathcal{O}_X) \\
\downarrow 0 & & \downarrow 0 \\
K_p(X \text{ on } \eta) & \overset{0}{\cong} & \mathcal{K}_p(k(X)) \\
\downarrow 0 & & \downarrow 1 \\
\bigoplus_{x \in X^{(1)}} K_{p-1}(X \text{ on } x) & \overset{1}{\cong} & \bigoplus_{x \in X^{(1)}} \mathcal{K}_{p-1}(k(x)) \\
\downarrow 0 & & \downarrow 1 \\
\vdots & & \vdots \\
\vdots & & \vdots \\
\downarrow 0 & & \downarrow 1 \\
\bigoplus_{x \in X^{(p-1)}} K_1(X \text{ on } x) & \overset{p-1}{\cong} & \bigoplus_{x \in X^{(p-1)}} \mathcal{K}_1(k(x)) \\
\downarrow 0 & & \downarrow 1 \\
\bigoplus_{x \in X^{(p)}} K_0(X \text{ on } x) & \overset{p}{\cong} & \bigoplus_{x \in X^{(p)}} \mathcal{K}_0(k(x))
\end{array}
\tag{9.2.1}
$$

Thus, each differential in Quillen's version of the Gersten complex, besides the augmentation, changes the weight by 1, while the horizontal maps change the weights be increasing amounts.

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