Math 1552 Section 17
PracticeTest 2 Solutions

Part I: Conceptual Questions.

1. Give an example of a conditionally convergent series that is not absolutely convergent.

   Solution: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

2. Give an example of an alternating series that does not converge.

   Solution: $\sum_{n=1}^{\infty} (-1)^n$.

Part II: Finding Sums.

3. Find the sum of the series $\frac{1}{5^2} - \frac{1}{5^4} + \frac{1}{5^6} - \frac{1}{5^8} + ...$ if it converges.

   Solution: $\frac{1}{5^2} - \frac{1}{5^4} + \frac{1}{5^6} - \frac{1}{5^8} + ... = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{25^n} = -\sum_{n=0}^{\infty} \left(-\frac{1}{25}\right)^n + 1 = -\frac{1}{1+\frac{1}{25}} + 1 = \mathbf{\frac{1}{26}}$.

4. Integrate both sides of the geometric series formula to find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

   Solution: Performing the integration gives:

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

   The right hand side is the same as $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Now plug in $x = \frac{1}{2}$ to get

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \ln 2.$$

Part III: Determining Convergence.

5. Does the series $\sum_{n=0}^{\infty} \pi^{-n}$ converge? Why or why not?

   Solution: **Yes,** and the sum is $\frac{\pi}{\pi - 1}$ by the geometric series formula.

6. Does the series $\sum_{n=1}^{\infty} \frac{n^n}{n^{n^n}}$ converge? Why or why not?

Solution: **Yes,** by the root test; the nth root of the general term is $\frac{n}{n^n}$, which approaches zero as n approaches infinity.

7. Does the series $\sum_{n=1}^{\infty} \frac{(2n)!}{(2n+1)!}$ converge? Why or why not?

   Solution: **No,** the general term simplifies to $\frac{1}{2n+1}$, which makes the result obvious.

Part IV: Power Series.

8. Calculate the Taylor series for $f(x) = xe^x$ centered at $x = 1$.

   Solution: The derivatives are $f'(x) = xe^x + e^x, f''(x) = xe^x + 2e^x, ..., f^{(n)}(x) = xe^x + ne^x$, so we have $f^{(n)}(1) = (n+1)e$, and the Taylor series gives

$$\mathbf{xe^x = \sum_{n=0}^{\infty} \frac{(n+1)e}{n!}(x-1)^n}.$$

9. Find the center and radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n(\ln n)^2}$. Does it converge at the endpoints? Why or why not?

   Solution: The ratio test gives

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)(\ln(n+1))^2} \frac{n(\ln n)^2}{x^n} \right| = |x|$$

   So **the radius of convergence is 1. The series converges at $x = 1$ by the integral test and converges at $x = -1$ by the alternating series test.**

10. Suppose $f(x)$ is infinitely many times differentiable for all x and $f^{(n)}(0) = 1$ for all n. What is $f(x)$?

    Solution: $\mathbf{f(x) = e^x}$, by the formula for the Maclaurin series.