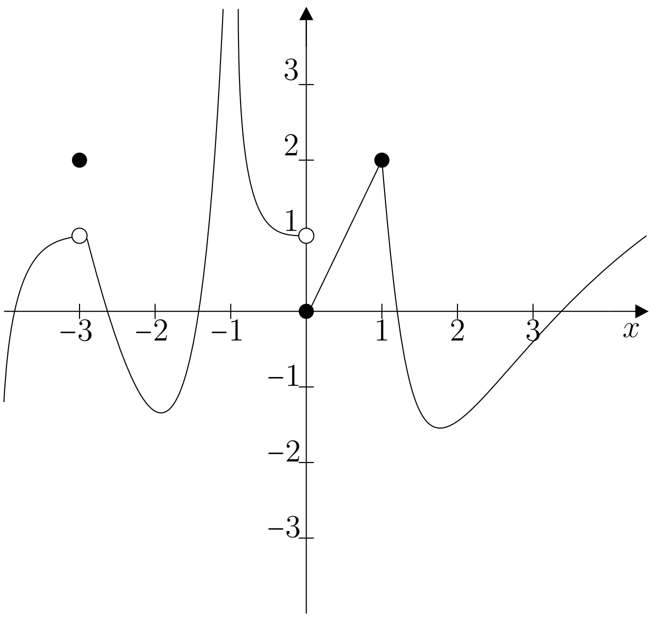
Math 1550 Section 20
Practice Final Exam Solutions

1. Consider the function $f(x)$ whose graph is shown below:



a) At which points does the limit of $f(x)$ not exist? Why?

Solution: $\mathbf{x = -1, 0}$. Infinite limit at -1, left and right limits disagree at 0.

b) Does $f(x)$ have asymptotes? If so, where are they?

Solution: Yes, vertical asymptote at $\mathbf{x = -1}$.

c) At which points is $f(x)$ discontinuous? Why?

Solution: $\mathbf{x = -3, -1, 0}$. At $-1, 0$, limit DNE. At -3, limit does not equal function value.

d) At which points is $f(x)$ non-differentiable? Why?

Solution: $\mathbf{x = -3, -1, 0, 1}$. At $-3, -1, 0$, discontinuous. At 1, no unique tangent line.

2. Find $\lim_{x \to 0} x^4 \sin \frac{1}{x}$, or state that the limit does not exist. Be sure to show your work!

Solution: $\mathbf{0}$, by the Squeeze Theorem, with $g(x) = -x^4, h(x) = x^4$.

3. State the definition of the derivative of $f(x)$.

Solution: $\mathbf{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$.

4. Find the equation of the tangent line to $f(x) = 3x$ at the point $(2, 6)$.

Solution: $\mathbf{y = 3x}$; it's its own tangent line.

5. What function is equal to its own derivative?

Solution: $\mathbf{f(x) = e^x}$.

6. A 15-foot ladder is leaning against a wall. At $t = 0$ seconds, the base of the ladder is 4 feet from the wall, and the top of the ladder begins to slide down the wall at 2 feet per second. How fast is the base of the ladder moving away from the wall at $t = 3$ seconds?

Solution: Call the distance from the wall to the base of the ladder x and the distance from the floor to the top of the ladder y.

Given: $\frac{dy}{dt} = -2$. Want: $\left.\frac{dx}{dt}\right|_{t=3}$. Equation relating x and y: $x^2 + y^2 = 15^2 = 225$.

Differentiate: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$.

Rearrange: $\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$.

Need values when $t = 3$. Now $x(0) = 4$, so $y(0) = \sqrt{209}$. So $y(3) = \sqrt{209} - 6$. So $x(3) = \sqrt{12\sqrt{209} - 20}$.

So $\left.\frac{dx}{dt}\right|_{t=3} = -\frac{\sqrt{209} - 6}{\sqrt{12\sqrt{209} - 20}}(-2)$. $= \mathbf{\frac{2\sqrt{209} - 12}{\sqrt{12\sqrt{209} - 20}}}$ feet per second.

7. Water flows into a cone-shaped cup at 2 cubic centimeters per second. The top of the cup has a radius of 5 centimeters, and the height of the cup is 10 centimeters. How fast is the level of the water in the cup rising when it is 1 centimeter from the top?

Solution: Call the volume of the water V and the height of the water h. Call the radius of the circular surface of the water r.

Given: $\frac{dV}{dt} = 2$. Want: $\frac{dh}{dt}\Big|_{h=9}$. Equation relating V and h: $V = \frac{1}{3}\text{base} \times \text{height} = \frac{1}{3}\pi r^2 h$.

Use similar triangles to eliminate r: $r/h = 5/10$, so $r = h/2$, so $V = \frac{1}{12}\pi h^3$.

Differentiate: $\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$. Rearrange: $\frac{4}{\pi h^2}\frac{dV}{dt} = \frac{dh}{dt}$. Plug in: $\frac{dh}{dt}\Big|_{h=9} = \frac{4}{81\pi}(2) = \mathbf{\frac{8}{81\pi}}$ cm/s.

8. Let $f(x) = x^3 - 2x^2 - x - 2$. Find the critical points, the maximum and minimum values, the points of inflection, and the intervals on which $f(x)$ is increasing, decreasing, concave up, and concave down, respectively. Sketch the graph of $f(x)$.

Solution: Too annoying to rehash all the details we did in class. The critical points are at $x = \frac{2}{3} \pm \frac{\sqrt{7}}{3}$. The inflection point is at $x = \frac{2}{3}$. The y-intercept is at $y = -2$. The x-intercept is around $x = 8/3$. You can take it from there!

9. What is the largest possible area that can be enclosed by placing two 20-foot pieces of fencing against a long wall (more than 40 feet long) to form a triangular enclosure?

Solution: In class, we found that the angle between the wall and each piece of fencing should be $\pi/4$, and the resulting area will be **200 square feet.**

10. State the Fundamental Theorem of Calculus.

Solution: $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x)dx} = \mathbf{F(b)} - \mathbf{F(a)}$, where $F(x)$ is any antiderivative of $f(x)$.

11. What is $\int_0^1 x^3 dx$?

Solution: $\mathbf{\frac{1}{4}}$.

12. Why is it necessary to put " plus C" in the answer to an indefinite integral?

Solution: Because "indefinite integral" means "antiderivative." Adding a constant to an antiderivative gives you another antiderivative because the constant is killed by differentiation. Hence, the answer is not unique.

13. Calculate $\frac{d}{dx} \int_0^x t^2 dt$.

Solution: $\mathbf{x^2}$; the derivative and integral "cancel."

14. Calculate $\int \cos^2(x) \sin(x) dx$.

Solution: Let $u = \cos(x)$. Then $du = -\sin(x)dx$, so $\sin(x)dx = -du$. Now substitute:

$$\int \cos^2(x) \sin(x) dx = -\int u^2 du = -\frac{u^3}{3} + C = -\mathbf{\frac{1}{3}\cos^3(x) + C}.$$

15. Calculate $\int \frac{e^x}{1 + e^x} dx$.

Solution: Let $u = 1 + e^x$. Then $du = e^x dx$, so

$$\int \frac{e^x}{1 + e^x} dx = \int \frac{du}{u} = \ln|u| + C = \mathbf{\ln(e^x + 1) + C}.$$

16. What is the area between the curves $f(x) = x^3$ and $f(x) = x^4$ from $x = 0$ to $x = 1$?

Solution: Area$= \int_0^1 (x^3 - x^4) dx = \frac{1}{4} - \frac{1}{5} = \mathbf{\frac{1}{20}}$.

17. What is the average value of the function $f(x) = \sin(x)$ from $x = 0$ to $x = \pi$?

Solution: Average $= \dfrac{\text{integral over interval}}{\text{length of interval}} = \dfrac{1}{\pi}\displaystyle\int_0^\pi \sin(x)dx = -\frac{1}{\pi}\cos(x)\Big|_0^\pi = -\frac{1}{\pi}\Big(-1-(1)\Big) = \mathbf{\frac{2}{\pi}}$.

18. Calculate the volume of the solid of revolution given by rotating the region between the x-axis and the graph of $f(x) = x^2$ from $x = 0$ to $x = 1$ about the x-axis.

Solution: Use disks: Volume $= \displaystyle\int_0^1 \pi(x^2)^2 dx = \pi\int_0^1 x^4 dx = \pi\frac{x^5}{5}\Big|_0^1 = \mathbf{\frac{\pi}{5}}$.

19. How much work does it take to stretch a spring 10 cm beyond its natural length if the spring constant is 100 newtons per meter?

Solution: Work $= \displaystyle\int (\text{force})d(\text{distance}) = \int_0^{\frac{1}{10}} 100x dx = 50x^2\Big|_0^{\frac{1}{10}} = \mathbf{\frac{1}{2}}$N. Note that centimeters must be converted to meters in the upper limit of the integral.

20. What is the arc length of the graph of the function $f(x) = x^{\frac{3}{2}}$ from $x = 0$ to $x = 2$?

Solution: Arc length $= \displaystyle\int_0^2 \sqrt{1 + \big(f'(x)\big)^2}dx = \int_0^2 \sqrt{1 + \Big(\frac{3}{2}x^{\frac{1}{2}}\Big)^2}dx = \int_0^2 \sqrt{1 + \frac{9}{4}x}dx.$

Let $u = 1 + \frac{9}{4}x$. Then $du = \frac{9}{4}dx$, so $dx = \frac{4}{9}du$. The limits of integration also change: when $x = 0$, $u = 1$, and when $x = 2$, $u = \frac{11}{2}$. Now substitute:

$$\int_0^2 \sqrt{1 + \frac{9}{4}x}dx = \frac{4}{9}\int_1^{\frac{11}{2}} u^{\frac{1}{2}}du = \left(\frac{4}{9}\right)\left(\frac{2}{3}\right)u^{\frac{3}{2}}\Big|_1^{\frac{11}{2}} = \mathbf{\frac{8}{27}\left(\left(\frac{11}{2}\right)^{\frac{3}{2}} - 1\right)}.$$