Math 1552 Section 16
Practice Final Exam

Part I: Techniques of Integration. Please compute the following integrals.

1. $\int x \ln x dx$

   Solution: Use integration by parts with $u = \ln x$ and $dv = xdx$. Then $du = (1/x)dx$ and $v = x^2/2$. Substituting into the formula $\int udv = uv - \int vdu$ gives you

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int xdx = \mathbf{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}.$$

2. $\int \sin^2\theta \cos\theta d\theta$

   Solution: Use integration by substitution with $u = \sin\theta$ and $du = \cos\theta d\theta$. This immediately gives you

$$\int \sin^2\theta\cos\theta d\theta = \int u^2 du = \frac{1}{3}u^3 + C = \mathbf{\frac{1}{3}\sin^3 x + C}.$$

3. $\int \frac{1}{(x+4)(x-1)}dx$

   Solution: Use partial fractions: there are two distinct linear factors, so

$$\frac{1}{(x+4)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$
$$1 = A(x-1) + B(x+4)$$

   Choose $x = 1$:

$$1 = 5B \text{ so } B = \frac{1}{5}$$

   Choose $x = -4$:

$$1 = -5A \text{ so } A = -\frac{1}{5}$$

   So $\frac{1}{(x+4)(x-1)} = -\frac{1}{5}\left(\frac{1}{x+4}\right) + \frac{1}{5}\left(\frac{1}{x-1}\right)$ and

$$\int \frac{1}{(x+4)(x-1)}dx = -\frac{1}{5}\int \frac{1}{x+4}dx + \frac{1}{5}\int \frac{1}{x-1}dx$$
$$= \mathbf{-\frac{1}{5}\ln|x+4| + \frac{1}{5}\ln|x-1| + C}$$

4. $\int_0^{\infty} xe^{-x^2} dx$

Solution: The meaning of the improper integral is

$$\int_0^{\infty} xe^{-x^2} dx = \lim_{R \to -\infty} \int_0^R xe^{-x^2} dx$$

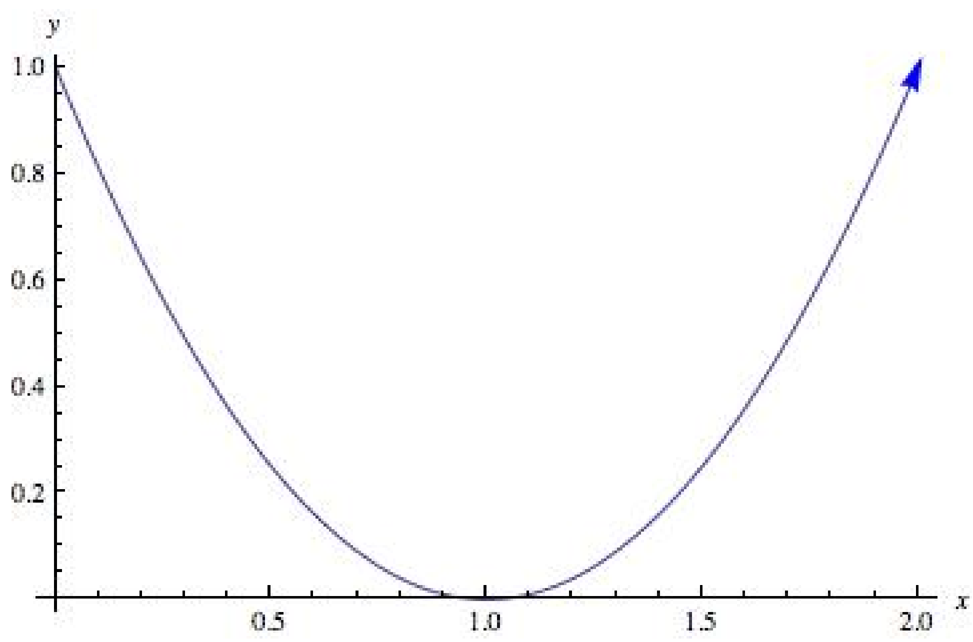
Let $u = x^2$, $du = 2xdx$, $xdx = \frac{1}{2}du$. Then the integral becomes

$$\begin{aligned} \lim_{R\to\infty} \frac{1}{2} \int_{x=0}^{R} e^{-u} du &= \lim_{R\to\infty} \left(-\frac{1}{2} e^{-u} \right)\Big|_{x=0}^{R} \\ &= \lim_{R\to\infty} \left(-\frac{1}{2} e^{-x^2} \right)\Big|_{x=0}^{R} \\ &= \lim_{R\to\infty} \left(-\frac{1}{2} e^{-R^2} + \frac{1}{2} e^0 \right) \\ &= \mathbf{\frac{1}{2}}. \end{aligned}$$

Part II: Parametric Equations.

5. Consider the parametric equations $x(t) = 1 + t$, $y(t) = t^2$, for $-1 \le t \le 1$. Sketch the parametric curve, including arrows to show motion along the curve as t increases. Solve for y in terms of x.

Solution: It's easiest to solve for y first: $t = x - 1$, so the equation $y = t^2$ becomes $\mathbf{y = (x-1)^2}$. Now the sketch is straightforward:



6. Use the arc length formula $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ to compute the arc length of the parametric curve $x(t) = \cos t$, $y(t) = \sin t$ from $t = 0$ to $t = \frac{\pi}{2}$.

Solution: The curve is just a quarter of the unit circle, so you know the answer should be $\pi/2$. Substituting $dx/dt = \sin t$ and $dy/dt = \cos t$ into the formula gives you

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\sin^2 t + \cos^2 t}\,dt = \int_0^{\frac{\pi}{2}} dt = t\Big|_0^{\frac{\pi}{2}} = \boldsymbol{\frac{\pi}{2}},$$

as expected.

Part III: Conic Sections.

7. Put the equation $x^2 + y^2 + 2x - 4y = 4$ into standard form by completing the square. Tell what kind of conic section it is, and make an accurate sketch.

Solution: First group x and y terms:

$$(x^2 + 2x) + (y^2 - 4y) = 4.$$

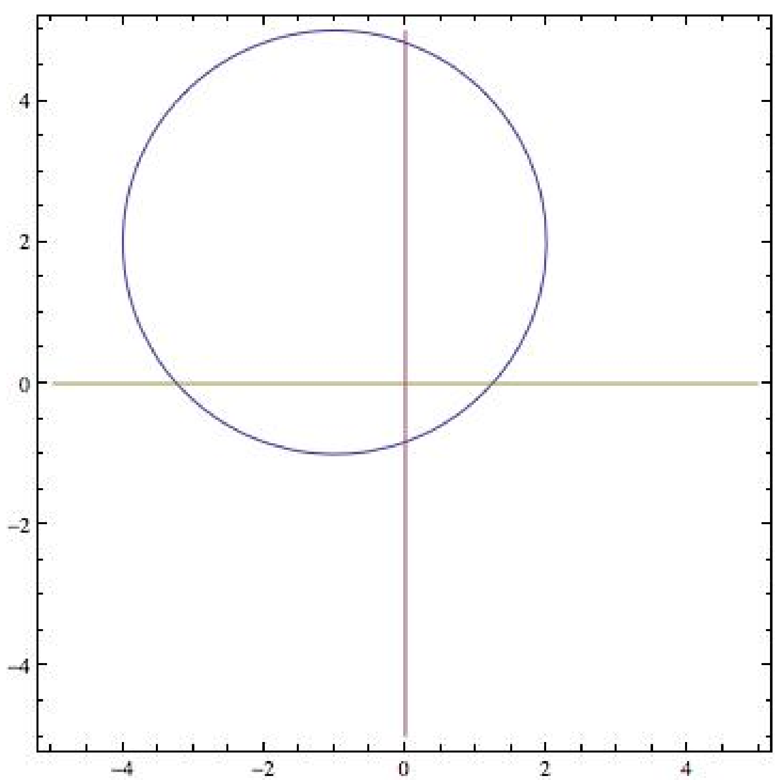
Now complete the square in each parenthesis and add the corresponding numbers on the right hand side:

$$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 4 + 1 + 4.$$

This simplifies to

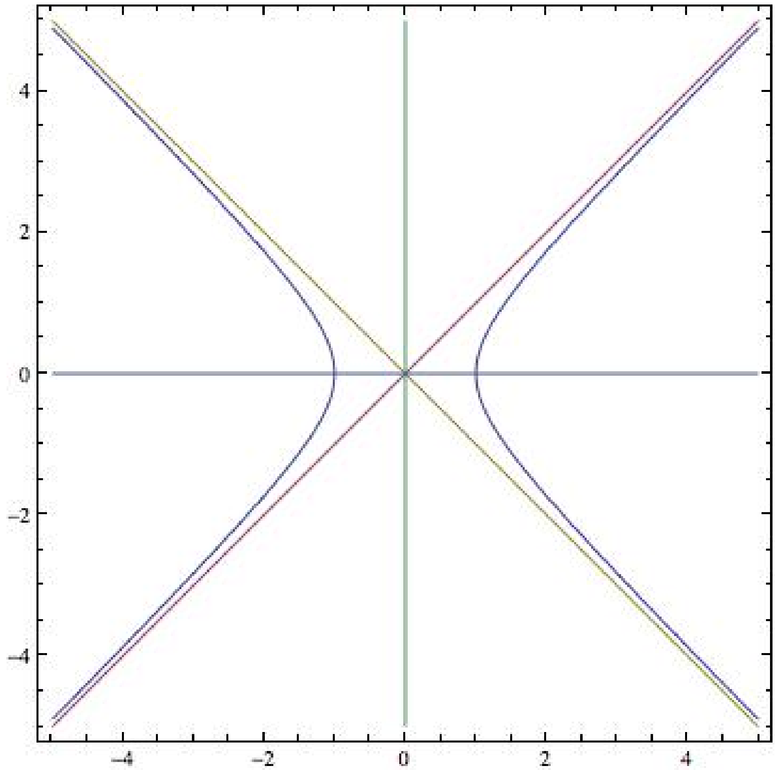
$$\mathbf{(x+1)^2 + (y-2)^2 = 3^2}.$$

So it's a **circle centered at** $(-1, 2)$**, with radius** 3**:**



8. Find the vertices and asymptotes of the hyperbola $x^2 - y^2 = 1$. Make an accurate sketch.

Solution: The asymptotes are given by "ignoring the constant," which means setting $x^2 = y^2$. This gives the two lines $\mathbf{y = \pm x}$. This means the hyperbola either has vertices on the x-axis or y-axis. Setting $x = 0$ in the equation $x^2 - y^2 = 1$ gives no solutions. Setting $y = 0$ gives the solutions $x = \pm 1$. So **the vertices are the points $(\pm 1, 0)$ on the x-axis.** Here's the sketch:



Part IV: Sequences and Series.

9. Find the sum of the series $\sum_{n=0}^{\infty} \frac{1}{4^n}$.

   Solution: Use the geometric series formula to get

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{1 - 1/4} = \mathbf{\frac{4}{3}}.$$

10. Match each series to the best test for determining whether or not it converges.

Solution: The colors indicate the best matches.

$\sum_{n=1}^{\infty} \frac{n!}{n^n}$	Comparison Test
$\sum_{n=1}^{\infty} \frac{n^2+1}{n^4}$	Limit Comparison Test
$\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{n^2+1}$	Root Test
$\sum_{n=1}^{\infty} \frac{1}{n \ln n}$	Alternating Series Test
$\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$	Divergence Test
$\sum_{n=1}^{\infty} \frac{3^n}{2^{2n}}$	Integral Test
$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{n^4}$	Ratio Test

11. Use the integral test to determine whether or not the series $\sum_{n=1}^{\infty} ne^{-n}$ converges.

Solution: The corresponding integral is $\int_{x=1}^{\infty} xe^{-x}dx$. Use integration by parts with $u = x$ and $dv = e^{-x}dx$. The answer is $2/e$, which is finite, so the series **converges.**

12. Write the series $\frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} + \frac{25}{243} + \ldots$ in summation notation. Does it converge? How do you know?

Solution: The numerators are squares, and the denominators are powers of 3. The series is

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n}.$$

It **converges** by the ratio test. By a previous quiz question, you know the exact sum is $3/2$.

13. Find the first five terms of the Taylor series for $f(x) = \sin x$ centered at $x = \frac{\pi}{2}$.

Solution: The first five derivatives, beginning at 0, are $f^{(0)}(x) = \sin x$, $f^{(1)}(x) = \cos x$, $f^{(2)}(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x$. Substituting the value $\pi/2$ gives $f^{(0)}(\pi/2) =$

1, $f^{(1)}(\pi/2) = 0$, $f^{(2)}(\pi/2) = -1$, $f^{(3)}(\pi/2) = 0$, $f^{(4)}(\pi/2) = 1$. Substituting this into Taylor's formula gives

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)(x-\pi/2)^n}{n!} = \mathbf{1+0-\frac{1}{2}(x-\pi/2)^2+0+\frac{1}{24}(x-\pi/2)^4+\ldots}$$

14. Find the first four terms of the MacLaurin series for $f(x) = \ln(1+x)$.

Solution: The "clever way" is to use differentiation and integration of power series, together with the geometric series formula. First observe that

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n,$$

by the geometric series formula. Now integrate term-by-term to get

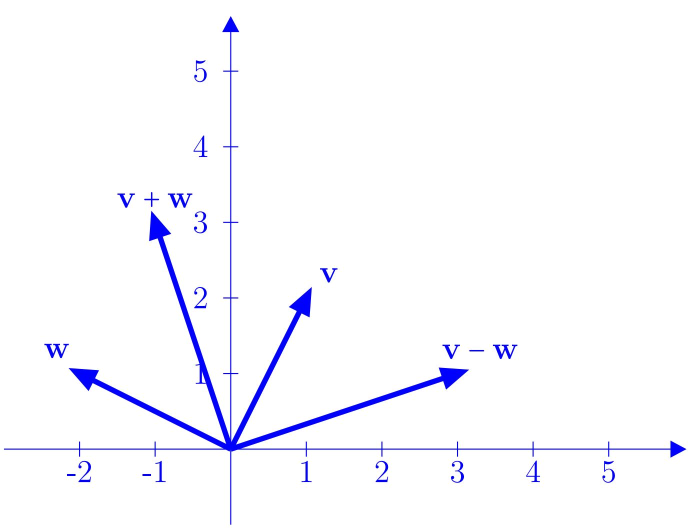
$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \mathbf{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots}$$

Part V: Vector Algebra.

15. Let $\mathbf{v} = \langle 1, 2\rangle$ and $\mathbf{w} = \langle -2, 1\rangle$ be vectors in the plane. Compute $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$. Make accurate sketches of $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v}+\mathbf{w}$, and $\mathbf{v}-\mathbf{w}$. Compute the lengths of all four vectors. Find unit vectors pointing in the directions of $\mathbf{v}$ and $\mathbf{w}$.

Solution: $\mathbf{v}+\mathbf{w} = \langle \mathbf{-1}, \mathbf{3}\rangle$, $\mathbf{v}-\mathbf{w} = \langle \mathbf{3}, \mathbf{1}\rangle$.

Sketches:



$\|\mathbf{v}\| = \sqrt{\mathbf{5}}$, $\|\mathbf{w}\| = \sqrt{\mathbf{5}}$, $\|\mathbf{v}+\mathbf{w}\| = \sqrt{\mathbf{17}}$, $\|\mathbf{v}-\mathbf{w}\| = \sqrt{\mathbf{17}}$.

Unit vectors: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \dfrac{\mathbf{1}}{\sqrt{\mathbf{5}}}, \dfrac{\mathbf{2}}{\sqrt{\mathbf{5}}}\right\rangle$, $\dfrac{\mathbf{w}}{\|\mathbf{w}\|} = \left\langle \dfrac{\mathbf{-2}}{\sqrt{\mathbf{5}}}, \dfrac{\mathbf{1}}{\sqrt{\mathbf{5}}}\right\rangle$.

16. Find the dot product of the vectors $\mathbf{v} = \langle 1, 1, 1\rangle$ and $\mathbf{w} = \langle 2, -1, -1\rangle$. What is the angle between $\mathbf{v}$ and $\mathbf{w}$? What is the component of $\mathbf{w}$ parallel to $\mathbf{v}$?

Solution: $\mathbf{v}\cdot\mathbf{w} = \mathbf{0}$, $\theta = \cos^{-1}\mathbf{0} = \pi/\mathbf{2}$, $\text{comp}_{\mathbf{v}}\mathbf{w} = \mathbf{0}$, $\text{comp}_{\mathbf{w}}\mathbf{v} = \mathbf{0}$.

17. Find two different unit vectors orthogonal to both of the vectors $\mathbf{v} = \langle 2, 0, 0\rangle$ and $\mathbf{w} = \langle 0, 2, 0\rangle$.

Solution: $\langle \mathbf{0}, \mathbf{0}, \pm\mathbf{1}\rangle$.

Part VI: Three-Dimensional Geometry.

18. Find an equation for the plane containing the point $(0, 1, 0)$, with normal vector $\langle 1, 0, 0\rangle$.

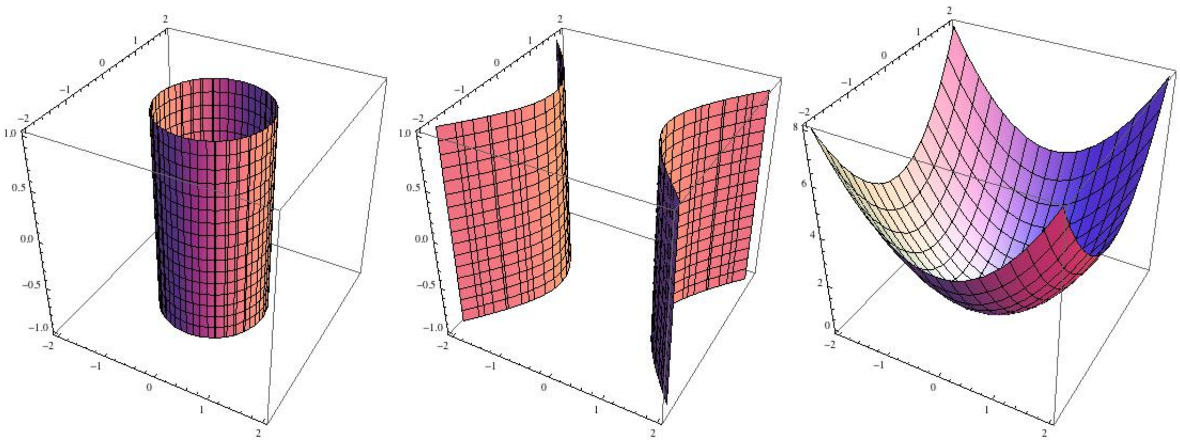
Solution: The plane equation $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ gives you

$$\langle 1, 0, 0\rangle \cdot \langle x - 0, y - 1, z - 0\rangle = 0,$$

which reduces to $\mathbf{x} = \mathbf{0}$.

19. Sketch and describe the surfaces defined by the equations $x^2 + y^2 = 1$, $x^2 - y^2 = 1$, and $x^2 + y^2 = z$.

Solution: The surfaces are a **circular cylinder of radius 1 symmetric about the z-axis,** a **hyperbolic cylinder whose trace in the xy-plane has asymptotes $y = \pm x$ and vertices $(\pm 1, 0)$,** and a **circular paraboloid symmetric about the z-axis, opening upward.** Here are the sketches:



Part VII: Partial Derivatives.

20. Let $f(x, y) = x^2 y + 2y^3$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. Show that $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

Solution: The first partial derivatives are

$$\frac{\partial f}{\partial x} = \mathbf{2xy}, \quad \frac{\partial f}{\partial y} = \mathbf{x^2 + 6y}.$$

The mixed second partials are

$$\frac{\partial^2 f}{\partial y \partial x} = \mathbf{2x}, \quad \frac{\partial^2 f}{\partial x \partial y} = \mathbf{2x}.$$