Math 1550 Section 20
Test 4 Solutions

1. Calculate $\int_0^1 100x^{99} dx$.

Solution: $\mathbf{1}$.

2. Calculate $\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx$. (Hint: $x^2 + y^2 = R^2$ is the equation of a circle centered at the origin with radius R.)

Solution: The integral is half the area of a circle of radius R, which is $\mathbf{\frac{1}{2}\pi R^2}$.

3. Calculate $\int_1^{e^2} \frac{d}{dx} \ln x dx$.

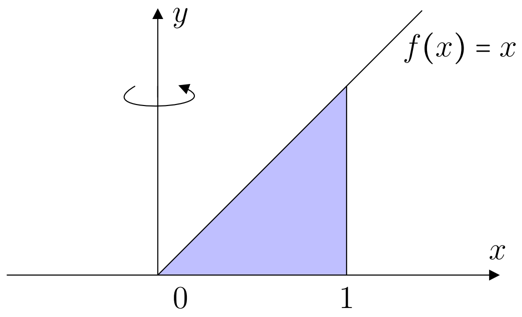
Solution: Since the antiderivative is $\ln x$, the FTC says

$$\int_1^{e^2} \frac{d}{dx} \ln x dx = \ln e^2 - \ln 1 = 2 \ln e - 0 = \mathbf{2}.$$

4. Calculate $\frac{d}{dx} \int_0^x \cos t dt$.

Solution: $\cos \mathbf{x}$, by the FTC.

5. Compute the volume of the solid of revolution given by revolving the region between the graph of $f(x) = x$ and the x-axis from $x = 0$ to $x = 1$ around the y-axis, using the method of washers.



Solution: Moving up the y-axis, the inner radius is y and the outer radius is 1. The limits of integration are $y = 0$ and $y = 1$. Thus, the volume integral is

$$V = \int_0^1 \pi(1 - y^2)dy = \pi\left(y - \frac{y^3}{3}\right)\Big|_0^1 = \mathbf{\frac{2\pi}{3}}.$$

6. Calculate $\int \frac{1}{\sin x \cos x} dx$. (Hint: First rewrite the integrand in terms of $\sec x$ and $\tan x$.)

Solution: Rewrite the integrand:

$$\frac{1}{\sin x \cos x} = \frac{\cos x}{\sin x}\frac{1}{\cos^2 x} = \frac{\sec^2 x}{\tan x}.$$

Now let $u = \tan x$. Then $du = \sec^2 x dx$, so the integral is

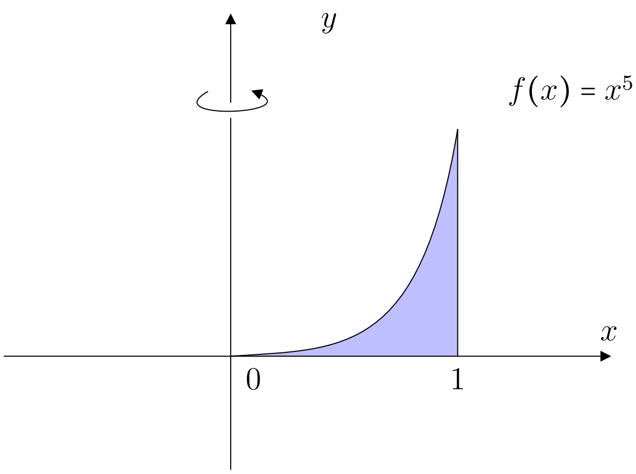
$$\int \frac{1}{\sin x \cos x} dx = \int \frac{\sec^2 x}{\tan x} dx = \int \frac{du}{u} = \ln|u| + C = \ln|\tan \mathbf{x}| + \mathbf{C}.$$

7. Calculate $\int \frac{xe^{x^2}}{1+e^{2x^2}}dx$ $\left(\text{Hint: } \frac{d}{du}\tan^{-1}(u) = \frac{1}{1+u^2}\right)$.

Solution: Let $u = e^{x^2}$. Then $du = 2xe^{x^2}dx$, so the integral is

$$\int \frac{xe^{x^2}}{1+e^{2x^2}}dx = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \mathbf{\frac{1}{2}\tan^{-1}e^{x^2} + C}.$$

8. Calculate the volume of the solid obtained by revolving the region between the graph of $f(x) = x^5$ and the x-axis from 0 to 1 around the y-axis, using the method of cylindrical shells.



Solution: The radius of a representative shell is x, and the height is $f(x) = x^5$. So the volume integral is

$$V = \int_0^1 (2\pi)(x)(x^5)dx = 2\pi\int_0^1 x^6 dx = 2\pi\frac{x^7}{7}\Big|_0^1 = \mathbf{\frac{2\pi}{7}}.$$

BONUS: Calculate $\int_0^\infty e^{-x}dx$ by replacing ∞ with a variable R in the upper limit of integration, and taking the limit of the resulting expression as $R \to \infty$. This is called an *improper integral.*

Solution:

$$\int_0^\infty e^{-x}dx = \lim_{R\to\infty}\int_0^R e^{-x}dx = \lim_{R\to\infty}\left(-e^{-x}\right)\Big|_0^R = \lim_{R\to\infty}\left(1-e^{-R}\right) = \mathbf{1}.$$