Math 1550 Section 20
Practice Test 2 Solutions

Solutions are in red. **Answers** are in blue.

1. Compute the derivative of $x^2x^4x^{10}$ using the product rule. Show all your steps.

Solution: There are several different ways you could break this into products, such as $(x^2x^4)(x^{10})$ and $(x^2)(x^4x^{10})$. Choosing the first method:

$$f'(x) = (x^2x^4)'(x^{10}) + (x^2x^4)(x^{10})'$$

$$= \Big((x^2)'(x^4) + (x^2)(x^4)'\Big)(x^{10}) + (x^2x^4)(x^{10})'$$

$$= \Big((2x)(x^4) + (x^2)(4x^3)\Big)(x^{10}) + (x^2x^4)(10x^9)$$

$$= (2x^5 + 4x^5)(x^{10}) + 10x^{15}$$

$$= \mathbf{16x^{15}}.$$

2. Let $f(x) = \dfrac{x^2+5x+6}{(x+1)e^x}$. Compute $f'(x)$.

Solution: This requires the quotient rule and the product rule:

$$f'(x) = \frac{(x+1)e^x\frac{d}{dx}\big(x^2+5x+6\big) - (x^2+5x+6)\frac{d}{dx}\big((x+1)e^x\big)}{\big((x+1)e^x\big)^2}$$

$$= \frac{\mathbf{(x+1)e^x(2x+5) - (x^2+5x+6)\big(e^x + (x+1)e^x\big)}}{\mathbf{\big((x+1)e^x\big)^2}}.$$

3. What is the 58th derivative of e^{2x}?

Solution: $\mathbf{2^{58}e^{2x}}$.

4. Write a general formula for the derivative of $f\big(g(x)h(x)\big)$ using the chain and product rules.

Solution: $\mathbf{f'\big(g(x)h(x)\big)\big(g'(x)h(x) + g(x)h'(x)\big)}$.

5. Let $f(x) = \sin\left(e^{\sin x} + \sin(e^x)\right)$. Find $f'(x)$.

Solution: $\mathbf{f'(x) = \cos\left(e^{\sin x} + \sin(e^x)\right)\left(e^{\sin x}\cos x + \cos(e^x)e^x\right)}$.

6. Let $f(x) = \left(\cos x\right)^{\cos x}$. Find $f'(x)$.

Solution: Use logarithmic differentiation: $\ln\left(f(x)\right) = \ln\left(\left(\cos x\right)^{\cos x}\right) = \cos x \ln\left(\cos x\right)$, so

$$\frac{d}{dx}\ln\left(f(x)\right) = \frac{f'(x)}{f(x)} = \frac{d}{dx}\cos x \ln\left(\cos x\right) = (-\sin x)\ln\left(\cos x\right) + \cos x\left(\frac{1}{\cos x}(-\sin x)\right)$$

$$= -\sin x\left(1 + \ln\left(\cos x\right)\right)$$

So $\mathbf{f'(x) = -\left(\cos x\right)^{\cos x}\sin x\left(1 + \ln\left(\cos x\right)\right)}$.

7. Suppose $x^y = \ln x$. Find $\frac{dy}{dx}$.

Solution: Take logarithms: $\ln\left(x^y\right) = y\ln x = \ln(\ln x)$. Now divide through by $\ln x$:

$$y = \frac{\ln(\ln x)}{\ln x}$$

This is an ordinary differentiation problem. Use the quotient rule and chain rule:

$$\frac{dy}{dx} = \frac{\ln x \frac{d}{dx}\left(\ln(\ln x)\right) - \ln(\ln x)\frac{d}{dx}\left(\ln x\right)}{\left(\ln x\right)^2} = \frac{\ln x\left(\frac{1}{\ln x}\frac{1}{x}\right) - \ln\left(\ln x\right)\left(\frac{1}{x}\right)}{\left(\ln x\right)^2}$$

$$= \mathbf{\frac{1 - \ln\left(\ln x\right)}{x\left(\ln x\right)^2}}.$$

8. The edge length of a cube is increasing at 2 cm/s. If the volume of the cube is 8 cm^3 when $t = 0$ seconds, how fast is the volume increasing when $t = 3$ seconds?

Solution: Call the edge length l and the volume V.

Given: $\frac{dl}{dt} = 2$ cm/s. Want: $\frac{dV}{dt}\Big|_{t=3}$. The equation relating l and V is $V = l^3$.

Differentiate using the chain rule: $\frac{dV}{dt} = 3l^2\frac{dl}{dt}$.

To solve this, we need the value of l when $t = 3$. Now $V(0) = 8\ \text{cm}^3$, so $l(0) = 2$ cm. Since l increases by 2 cm every second, $l(3) = 2 + 3(2) = 8$ cm. Thus,

$$\frac{dV}{dt}\Big|_{t=3} = 3(8\text{ cm})^2(2\text{ cm/s}) = \mathbf{384}\ \text{cm}^3/\text{s}$$

9. The volume of a spherically-shaped boy is given by the function $8+3(1-e^{-t})$, measured in cubic feet. How fast is his radius increasing when his volume is 10 cubic feet?

Solution: Here we are given the volume in terms of time, and have to calculate $\frac{dV}{dt}$.

Calculate: $\frac{dV}{dt} = 3e^{-t}\ \text{cm}^3/\text{s}$. Want: $\frac{dr}{dt}\Big|_{V=10}$. The equation relating r and V is $V = \frac{4}{3}\pi r^3$.

Differentiate using the chain rule: $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. Solve to get $\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$.

We need r and $\frac{dV}{dt}$ when $V = 10\ \text{cm}^3$. To get r, solve the equation $V = \frac{4}{3}\pi r^3$ for r when $V = 10$ to get $r = \left(\frac{15}{2\pi}\right)^{\frac{1}{3}}$. To get $\frac{dV}{dt}\Big|_{V=10}$, first solve for t:

$$10 = 8+3(1-e^{-t}) \quad \text{so} \quad e^{-t} = \frac{1}{3} \quad \text{so} \quad t = \ln 3 \quad \text{when} \quad V = 10.$$

So $\frac{dV}{dt}\Big|_{V=10} = 3e^{-\ln 3} = 1$. Therefore, $\mathbf{\frac{dr}{dt}\Big|_{V=10} = \frac{1}{4\pi}\left(\frac{2\pi}{15}\right)^{\frac{2}{3}}}$ cm/s.

BONUS: Find the derivative of x^{x^x}.

Solution: First find the derivative of $g(x) = x^x$ using logarithmic differentiation:

$$\frac{g'(x)}{g(x)} = \frac{d}{dx}x\ln x = \ln x + 1 \quad \text{so} \quad g'(x) = x^x(\ln x + 1).$$

First find the derivative of $f(x) = x^{g(x)} = x^{x^x}$ using logarithmic differentiation:

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}g(x)\ln x = \left(x^x(\ln x+1)\right)\ln x + x^x\frac{1}{x} \quad \text{so} \quad \mathbf{f'(x) = x^{x^x}\left(\left(x^x(\ln x+1)\right)\ln x + x^x\frac{1}{x}\right)}.$$