Math 1552 Section 16
Test 2 Solutions

Part I: Conic Sections.

1. Put the equation $x^2+3y^2+2x-12y+10=0$ into standard form. Tell what kind of conic section it is. If it is a parabola, give the focus and directrix. If it is an ellipse, tell where it is centered, and give the major and minor axes and foci. If it is a hyperbola, tell where it is centered, and give the vertices and asymptotes. Make an accurate sketch.

   Solution: Complete the square:

   $$x^2+2x+1+3(y^2-4y+4)=-10+1+12, \quad \text{so} \quad \mathbf{\frac{(x+1)^2}{3}+(y-2)^2=1}.$$

   It's an **ellipse.** The center is at $\mathbf{(-1,2)}$. The major axis is $\mathbf{2\sqrt{3}}$, and the minor axis is $\mathbf{2}$. The foci are at $\mathbf{(-1\pm\sqrt{2},2)}$.

   I'll draw it in class.

2. Give an equation for a parabola with focus $(1,0)$ and directrix $x=-1$. Make an accurate sketch.

   Solution: The trick is that it opens in the y-direction, not the x-direction. So the equation is $x=\frac{1}{4p}y^2$ with $p=1$, or $\mathbf{x=\frac{1}{4}y^2}$.

   I'll draw it in class.

3. Find the vertices and asymptotes of the hyperbola $x^2-4y^2=16$. Make an accurate sketch.

   Solution: This one was straight from the practice test. The standard form of the equation is

   $$\frac{x^2}{16}-\frac{y^2}{4}=1.$$

   Thus, the vertices are at $\mathbf{(\pm 4,0)}$, and the asymptotes are $\mathbf{y=\pm x/2}$.

   I'll draw it in class.

Part II: Sequences and Series.

3. Use the ratio test to determine whether the series $\sum_{n=2}^{\infty}\frac{1}{\ln(n!)}$ converges.

   Solution: Besides having two number 3's, I graded this one wrong! **The ratio test is inconclusive.** The limit you get is $\lim_{n\to\infty}\left|\frac{\ln(n!)}{\ln((n+1)!)}\right|$. The logarithm of a product is the sum of the logarithms of the factors, so this becomes

$$\lim_{n\to\infty}\left|\frac{\ln(n!)}{\ln((n+1)!)}\right| = \lim_{n\to\infty}\left|\frac{\ln(n!)}{\ln(n+1)+\ln(n!)}\right| = 1.$$

By the way, **the series diverges.** The easiest way to see this is to compare it to the series $\sum_{n=2}^{\infty}\frac{1}{\ln(n^n)} = \sum_{n=2}^{\infty}\frac{1}{n\ln n}$, which diverges by the integral test.

4. Find the sum of the series $\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n}$.

   Solution: This is the sum of two geometric series:

$$\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n} = \sum_{n=1}^{\infty}\frac{2^n}{4^n} + \sum_{n=1}^{\infty}\frac{3^n}{4^n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty}\frac{3}{4}\left(\frac{3}{4}\right)^n$$
$$= \frac{1/2}{1-1/2} + \frac{3/4}{1-3/4} = \mathbf{4}$$

5. Use the integral test to determine whether or not the series $\sum_{n=1}^{\infty} ne^{-n}$ converges.

   Solution: Use integration by parts:

$$\int_{x=1}^{\infty} xe^{-x}dx = \lim_{R\to\infty}\int_{x=1}^{R} xe^{-x}dx = \lim_{R\to\infty}\left(-xe^{-x}\Big|_1^R + \int_{x=1}^{R} e^{-x}dx\right) = \lim_{R\to\infty}\left(-xe^{-x}-e^{-x}\right)\Big|_1^R = \frac{2}{e} < \infty.$$

   So **the series converges** by the integral test.

6. Is the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^5 6^n}{n!}$ absolutely convergent, conditionally convergent, or divergent? Explain your reasoning.

   Solution: The series converges absolutely by the ratio test. The limit you get is

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)^5 6^{n+1}}{(n+1)!}\frac{n!}{(-1)^{n+1}n^5 6^n}\right| = \lim_{n\to\infty}\frac{6(n+1)^5}{(n+1)}\frac{1}{n^5} = 6\lim_{n\to\infty}\frac{(n+1)^4}{(n^5)} = 0.$$

7. Match each series to the best test for determining whether or not it converges.

Solution: The colors indicate the best matches.

$\sum_{n=1}^{\infty} \frac{n!}{n^n}$	**Comparison Test**
$\sum_{n=1}^{\infty} \frac{n^2+1}{n^4}$	**Limit Comparison Test**
$\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{n^2+1}$	**Root Test**
$\sum_{n=1}^{\infty} \frac{1}{n \ln n}$	**Alternating Series Test**
$\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$	**Divergence Test**
$\sum_{n=1}^{\infty} \frac{3^n}{2^{2n}}$	**Integral Test**
$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{n^4}$	**Ratio Test**

8. Does the series $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$ converge? Why or why not?

Solution: This was straight from the practice test. **The series converges by the ratio test.** The limit you get is

$$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}/((n+1)!)^2}{n^n/(n!)^2} \right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{((n+1)!)^2} \frac{(n!)^2}{n^n} = \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{(n+1)^2(n!)^2} \frac{(n!)^2}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n+1} \frac{1}{n^n} = \lim_{n\to\infty} \frac{1}{n+1} \left(\frac{n+1}{n} \right)^n.$$

So far, this is just algebraic rearranging. The potential sticking point is to remember from Calculus I that

$$\lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n = e.$$

From this it follows immediately that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{(n+1)}\left(\frac{n+1}{n}\right)^n = e\lim_{n\to\infty}\frac{1}{n+1} = \mathbf{0}.$$

Since the limit is less than 1, the ratio test guarantees convergence.

9. Use the root test to show that the series $\sum_{n=1}^{\infty}\frac{2^n}{n^n}$ converges.

Solution: Again, straight from the practice test. The limit you get is $\lim_{n\to\infty}\frac{2}{n} = \mathbf{0}$. Since the limit is less than 1, the root test guarantees convergence.

10. Write the series $\frac{1}{2}+\frac{4}{4}+\frac{9}{8}+\frac{16}{16}+\frac{25}{32}+\ldots$ in summation notation. Does it converge? How do you know?

Solution: The numerators are $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$ and the denominators are $2^1, 2^2, 2^3, 2^4, 2^5, \ldots$, so the series is $\boldsymbol{\sum_{n=1}^{\infty}\frac{n^2}{2^n}}$. It **converges** because the exponential in the denominator dominates the power in the numerator. You can show this precisely using the ratio test.

11. BONUS 1: Find the sum of the series $\sum_{n=1}^{\infty}\frac{n}{2^n}$.

Solution: First way:

$$\sum_{n=1}^{\infty}\frac{n}{2^n} = \sum_{n=1}^{\infty}\frac{1}{2^n}+\sum_{n=2}^{\infty}\frac{1}{2^n}+\sum_{n=3}^{\infty}\frac{1}{2^n}+\ldots = 1+\frac{1}{2}+\frac{1}{4}+\ldots = \sum_{n=0}^{\infty}\frac{1}{2^n} = \mathbf{2}.$$

Second way: differentiate the formula $\sum_{n=0}^{\infty}ar^n = \frac{a}{1-r}$ with respect to r:

$$\frac{d}{dr}\sum_{n=0}^{\infty}ar^n = \frac{d}{dr}\frac{a}{1-r}.$$

The left-hand side is $\sum_{n=0}^{\infty}anr^{n-1}$. Substitute $a = 1$, $r = 1/2$. The first term is zero, so you get $\sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^{n-1} = 2\sum_{n=1}^{\infty}\frac{n}{2^n}$. The right-hand side is $\frac{a}{(1-r)^2}$, which comes out to 4 using the same substitutions. Dividing both sides by 2,

$$\sum_{n=1}^{\infty}\frac{n}{2^n} = \mathbf{2}.$$

12. BONUS 2: Does the series $\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{(n!)}}$ converge?

Solution: **The series converges by the ratio test.** The limit you get is

$$\lim_{n\to\infty}\left|\frac{((n+1)!)^{n+1}}{(n+1)^{(n+1)!}}\frac{n^{(n!)}}{(n!)^n}\right| = \lim_{n\to\infty}\frac{(n+1)!((n+1)!)^n}{(n+1)^{(n+1)!}}\frac{n^{(n!)}}{(n!)^n} = \lim_{n\to\infty}\frac{(n+1)!(n+1)^n(n!)^n}{(n+1)^{(n+1)!}}\frac{n^{(n!)}}{(n!)^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)!n^{(n!)}}{(n+1)^{(n+1)!-n}} = \lim_{n\to\infty} \left(\frac{(n+1)!}{(n+1)^{(n+1)}}\right)\left(\frac{n^{(n!)}}{(n+1)^{(n+1)!-2n-1}}\right) = 0,$$

since both fractions in the limit go to zero separately.