Math 1552 Section 16
Test 1 Solutions

Part I: Techniques of Integration. Please compute the following integrals.

1. $\int_e^{e^2} x\ln(x^4)dx$

Solution: Integration by parts: $u = \ln(x^4)$, $v' = x$, $u' = \frac{1}{x^4}\cdot 4x^3 = \frac{4}{x}$, $v = \frac{1}{2}x^2$. Then:

$$\begin{aligned}\int_e^{e^2} x\ln(x^4)dx &= \frac{1}{2}x^2\ln(x^4)\Big|_e^{e^2} - \int_e^{e^2}\frac{1}{2}x^2\cdot\frac{4}{x}dx \\ &= 2x^2\ln x\Big|_e^{e^2} - 2\int_e^{e^2} xdx \\ &= (2x^2\ln x - x^2)\Big|_e^{e^2} \\ &= (2e^4\ln(e^2) - e^4) - (2e^2\ln(e) - e^2) \\ &= (4e^4 - e^4) - (2e^2 - e^2) \\ &= \mathbf{3e^4 - e^2}\end{aligned}$$

2. $\int \frac{x+1}{2x^2+1}dx$

Solution: It's easiest to split the integral into two parts:

$$\int\frac{x+1}{2x^2+1}dx = \int\frac{x}{2x^2+1}dx + \int\frac{1}{2x^2+1}dx.$$

For the first integral, substitute $u = 2x^2+1$, $du = 4xdx$, so $xdx = \frac{1}{4}du$. This gives

$$\int\frac{x}{2x^2+1}dx = \frac{1}{4}\int\frac{1}{u}du = \frac{1}{4}\ln(2x^2+1) + C.$$

For the second integral, substitute $x = \frac{1}{\sqrt{2}}\tan\theta$. Note that the inverse of this is $\theta = \tan^{-1}(\sqrt{2}x)$. Also, $2x^2+1 = \tan^2\theta + 1 = \sec^2\theta$. Then $dx = \frac{1}{\sqrt{2}}\sec^2\theta$. So

$$\int\frac{1}{2x^2+1}dx = \int\frac{1}{\sec^2\theta}\frac{1}{\sqrt{2}}\sec^2\theta = \frac{1}{\sqrt{2}}\int d\theta = \frac{1}{\sqrt{2}}\theta + C = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x) + C.$$

Combining these,

$$\int\frac{x+1}{2x^2+1}dx = \mathbf{\frac{1}{4}\ln(2x^2+1) + \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x) + C}.$$

3. $\int_0^{\frac{\pi}{4}} \sin^2\theta\cos^2\theta d\theta$

Solution:

$$\int_0^{\frac{\pi}{4}} \sin^2 x\cos^2 x dx = \frac{1}{4}\int_0^{\frac{\pi}{4}} \sin^2 2x dx = \frac{1}{8}\int_0^{\frac{\pi}{4}} (1-\cos 4x)dx = \left(\frac{1}{8}x - \frac{1}{32}\sin 4x\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{8}\frac{\pi}{4} - \frac{1}{32}\sin\pi\right) - \left(\frac{1}{8}0 - \frac{1}{32}\sin 0\right) = \frac{\pi}{\mathbf{32}}.$$

4. $\int \frac{1}{(x+4)(x-1)}dx$

Solution: Partial fractions: there are two distinct linear factors, so

$$\frac{1}{(x+4)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$
$$1 = A(x-1) + B(x+4)$$

Choose $x = 1$:

$$1 = 5B \text{ so } B = \frac{1}{5}$$

Choose $x = -4$:

$$1 = -5A \text{ so } A = -\frac{1}{5}$$

So $\frac{1}{(x+4)(x-1)} = -\frac{1}{5}\left(\frac{1}{x+4}\right) + \frac{1}{5}\left(\frac{1}{x-1}\right)$ and

$$\int \frac{1}{(x+4)(x-1)}dx = -\frac{1}{5}\int \frac{1}{x+4}dx + \frac{1}{5}\int \frac{1}{x-1}dx$$
$$= -\frac{\mathbf{1}}{\mathbf{5}}\ln|\mathbf{x}+\mathbf{4}| + \frac{\mathbf{1}}{\mathbf{5}}\ln|\mathbf{x}-\mathbf{1}| + \mathbf{C}$$

5. $\int \frac{7x^2+x+3}{(2x^2+1)(x+1)}dx$

Solution:

$$\frac{7x^2+x+3}{(2x^2+1)(x+1)} = \frac{Ax+B}{2x^2+1} + \frac{C}{x+1}$$

Multiply both sides by $(2x^2+1)(x+1)$:

$$7x^2+x+3 = (Ax+B)(x+1) + C(2x^2+1)$$

Set $x = -1$ to get

$$9 = 3C, \text{ so } C = 3$$

Set $x = 0$ to get

$$3 = B + 3, \text{ so } B = 0$$

Set $x = 1$ to get

$$11 = 2A + 9, \text{ so } A = 1$$

So

$$\int \frac{7x^2 + x + 3}{2x^3 + 2x^2 + x + 1} dx = \int \frac{x}{2x^2 + 1} dx + 3 \int \frac{1}{x+1} dx$$
$$= \mathbf{\frac{1}{4} \ln(2x^2 + 1) + 3 \ln|x + 1| + C}$$

(the first integral appears in problem 2).

6. $\int \sin^{-1} x dx$

Solution: Choose $u = \sin^{-1} x$, $du = \frac{1}{\sqrt{1-x^2}} dx$, $dv = dx$, $v = x$.

$$\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

On the second integral, use $w = 1 - x^2$, $dw = -2x dx$, $x dx = -\frac{1}{2} dw$. Then

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int w^{-\frac{1}{2}} dw$$
$$= -\frac{1}{2} \frac{w^{\frac{1}{2}}}{\frac{1}{2}}$$
$$= -\sqrt{w}$$
$$= -\sqrt{1-x^2}$$

Subtracting the second integral from the first:

$$\mathbf{x \sin^{-1} x + \sqrt{1 - x^2} + C}$$

7. $\int_{-\infty}^{\infty} xe^{-x^2}dx$ (Hint: break the integral into two pieces, one from $-\infty$ to 0, and the other from 0 to ∞.)

Solution: The integrand is an odd function, so you can immediately say the answer is zero. But here's the calculus anyway:

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx$$

$$= \lim_{R_1\to-\infty}\int_{R_1}^{0} xe^{-x^2}dx + \lim_{R_2\to\infty}\int_{0}^{R_2} xe^{-x^2}dx$$

Let $u = x^2$, $du = 2xdx$, $xdx = \frac{1}{2}du$. Then the integral becomes

$$\lim_{R_1\to-\infty}\frac{1}{2}\int_{x=R_1}^{0} e^{-u}du + \lim_{R_2\to\infty}\frac{1}{2}\int_{x=0}^{R_2} e^{-u}du$$

$$= \lim_{R_1\to-\infty}\left(-\frac{1}{2}e^{-u}\right)\Big|_{x=R_1}^{0} + \lim_{R_2\to\infty}\left(-\frac{1}{2}e^{-u}\right)\Big|_{x=0}^{R_2}$$

$$= \lim_{R_1\to-\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_{x=R_1}^{0} + \lim_{R_2\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_{x=0}^{R_2}$$

$$= \lim_{R_1\to-\infty}\left(-\frac{1}{2}e^{0} + \frac{1}{2}e^{-R_1^2}\right) + \lim_{R_2\to\infty}\left(-\frac{1}{2}e^{-R_2^2} + \frac{1}{2}e^{0}\right)$$

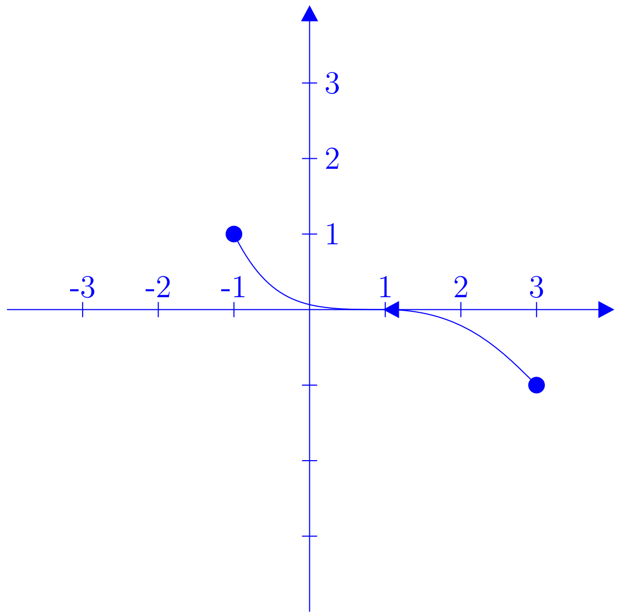
$$= -\frac{1}{2} + \frac{1}{2}$$

$$= \mathbf{0}$$

Part II: Parametric Equations.

8. Consider the parametric equations $x(t) = 1-2t$, $y(t) = t^3$, for $-1 \le t \le 1$. Sketch the parametric curve, including arrows to show motion along the curve as t increases. Solve for y in terms of x. Compute the derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution: If you solve the first equation for t, you get $t = (1-x)/2$, so $y = \frac{1}{8}(1-x)^3$. The graph looks like this:



The derivatives are

$$\mathbf{\frac{dy}{dx} = -\frac{3}{8}(1-x)^2},$$

and

$$\mathbf{\frac{d^2y}{dx^2} = \frac{3}{4}(1-x)}.$$

9. Compute the arc length of the parametric curve $x(t) = \sin t$, $y(t) = \dfrac{1}{2}\sin^2 t$ from $t = 0$ to $t = \dfrac{\pi}{2}$.

   Solution: This turned out to be harder than I had intended, too hard for a test. You can begin by substituting the derivatives

$$\frac{dx}{dt} = \cos t \quad \text{and} \quad \frac{dy}{dt} = \sin t \cos t$$

   into the arc length formula to get

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t + \sin^2 t \cos^2 t} dt = \int_0^{\frac{\pi}{2}} \cos t \sqrt{1 + \sin^2 t} dt$$

   If you then substitute $u = \sin t$, you get

$$L = \int_0^1 \sqrt{1+u^2} du.$$

   One possible approach is now to substitute $u = \tan\theta$ to get

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2\theta} \sec^2\theta d\theta = \int_0^{\frac{\pi}{4}} \sec^3\theta,$$

but this is still an annoying integral. If you recall, we actually wasted half an hour of class doing this integral, using integration by parts multiple times. A better solution, which we unfortunately didn't cover in class, is to make the *hyperbolic substitution* $u = \sinh\theta$, since then $\sqrt{1+u^2} = \sqrt{1+\sinh^2\theta} = \sqrt{\cosh^2\theta} = \cosh\theta$, and $du = \cosh\theta$. We then need to evaluate the integral

$$L = \int_{u=0}^{1} \cosh^2\theta d\theta,$$

which comes out to

$$L = \frac{1}{2}\Big(\theta + \sinh\theta\cosh\theta\Big)\Big|_{u=0}^{1},$$

or in terms of u,

$$L = \frac{1}{2}\Big(\sinh^{-1}u + u\sqrt{1+u^2}\Big)\Big|_{u=0}^{1} = \mathbf{\frac{1}{2}\Big(\sinh^{-1}1 + \sqrt{2}\Big)},$$

where $\sinh^{-1}1$ is about .881 in decimals.

10. Compute the surface area of the surface of revolution given by rotating the parametric curve $x(t) = \frac{t^2}{2}$, $y(t) = t$ from $t = 0$ to $t = 1$ about the x-axis.

Solution: Begin by substituting the derivatives

$$\frac{dx}{dt} = t \quad \text{and} \quad \frac{dy}{dt} = 1$$

into the surface area formula to get

$$A = \int_0^1 2\pi t\sqrt{t^2+1}dt.$$

Now substitute $u = t^2 + 1$, $du = 2tdt$ to get

$$A = \frac{1}{2}\int_1^2 2\pi\sqrt{u}du = \frac{2\pi}{3}u^{\frac{3}{2}}\Big|_1^2 = \mathbf{\frac{2\pi}{3}\Big(2^{\frac{3}{2}} - 1\Big)}.$$

BONUS:

11. Show that $\int_{\frac{\pi^2}{16}}^{\frac{\pi^2}{4}} \frac{1}{\sqrt{x}} e^{-\tan\sqrt{x}}\left(1+\tan^2\sqrt{x}\right)dx = \frac{2}{e}$

Solution:

$u = \tan\sqrt{x}$, $du = \frac{1}{2}\frac{\sec^2\sqrt{x}}{\sqrt{x}}dx = \frac{1}{2}\left(\frac{1+\tan^2\sqrt{x}}{\sqrt{x}}\right)dx$, so $\frac{1+\tan^2\sqrt{x}}{\sqrt{x}}dx = 2du$. Then

$$\int_{\frac{\pi^2}{16}}^{\frac{\pi^2}{4}} \frac{1}{\sqrt{x}} e^{-\tan\sqrt{x}}\left(1+\tan^2\sqrt{x}\right)dx = 2\int_{x=\frac{\pi^2}{16}}^{\frac{\pi^2}{4}} e^{-u}du$$

Now look at the limits. If $x = \frac{\pi^2}{16}$, then $u = \tan\frac{\pi}{4} = 1$, and if $x = \frac{\pi^2}{4}$, then $u = \tan\frac{\pi}{2} = +\infty$, so this is an improper integral. Written in terms of u, the integral is

$$2\int_{u=1}^{\infty} e^{-u}du$$
$$= 2\lim_{R\to\infty}\int_1^R e^{-u}du$$
$$= 2\lim_{R\to\infty}\left(-e^{-u}\right)\Big|_{u=1}^{R}$$
$$= 2\lim_{R\to\infty}\left(-e^{-R}+e^{1}\right)$$
$$= \mathbf{\frac{2}{e}}$$