Realizing Zero-Divisor Graphs

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Acknowledgments

This project is inspired by Dr. Sandra Spiroff from the University of Mississippi and is mentored by Benjamin Dribus from Louisiana State University.

Outline

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Additionally, R is **commutative if for all a, b in R, $a \cdot b = b \cdot a$; and R has **unity** if 1 is in R such that $a \cdot 1 = 1 \cdot a$ for all a in R.

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$$\begin{aligned} 2 \cdot 3 &= 6 &\equiv 0 \bmod 6 \\ 3 \cdot 4 &= 12 &\equiv 0 \bmod 6 \end{aligned}$$

Anderson-Livingston Graphs

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An **Anderson-Livingston zero-divisor graph** of a commutative ring, R, with unity is a simple graph (i.e. with no loops or multiple edges) whose set of vertices consists of all non-zero zero divisors, with an edge between a and b if $a \cdot b = 0$. These graphs will be denoted $\Gamma(R)$.

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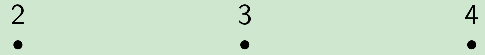

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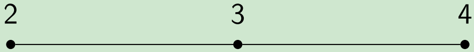

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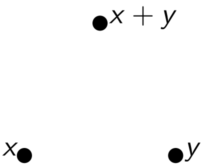

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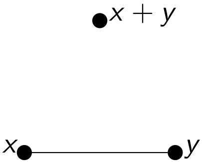

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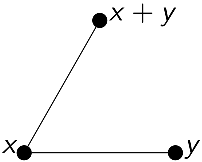

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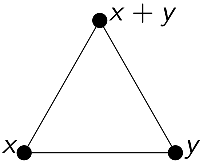

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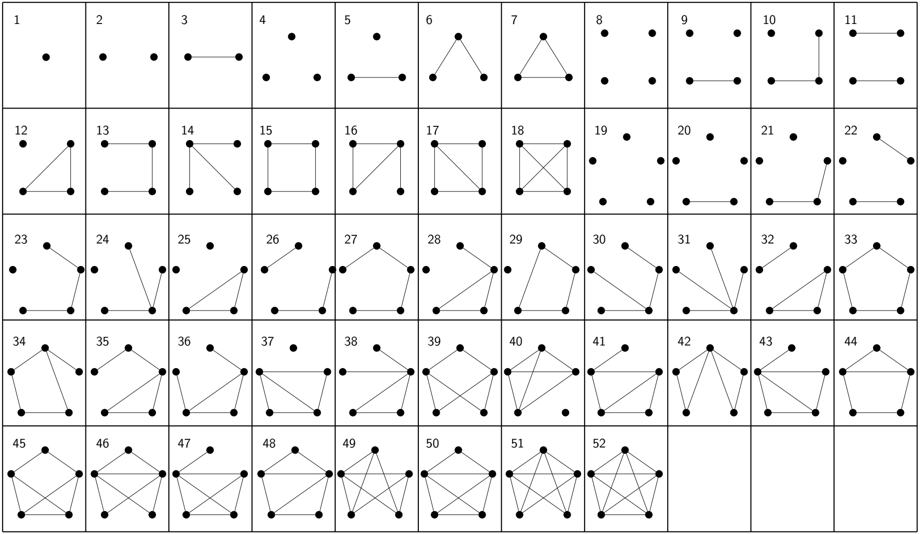
- Draw all the graphs on 1-5 vertices.
- Determine which of these graphs can be realized as the zero-divisor graph of a ring, R.
- Give examples of rings associated with these possible graphs.
- Provide proofs for graphs which cannot be realized as a zero-divisor graph of a ring.

The Graphs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The **diameter** of a graph, G, denoted $diam(G)$, is the greatest distance between two vertices (i.e. the maximal number of edges between two vertices).

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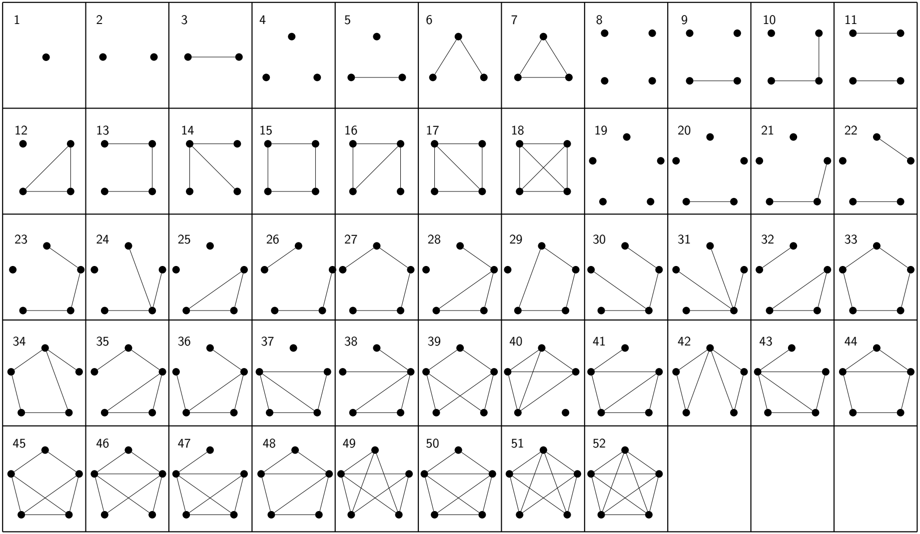
$diam(\Gamma(R)) \leq 3$

New Set of Graphs



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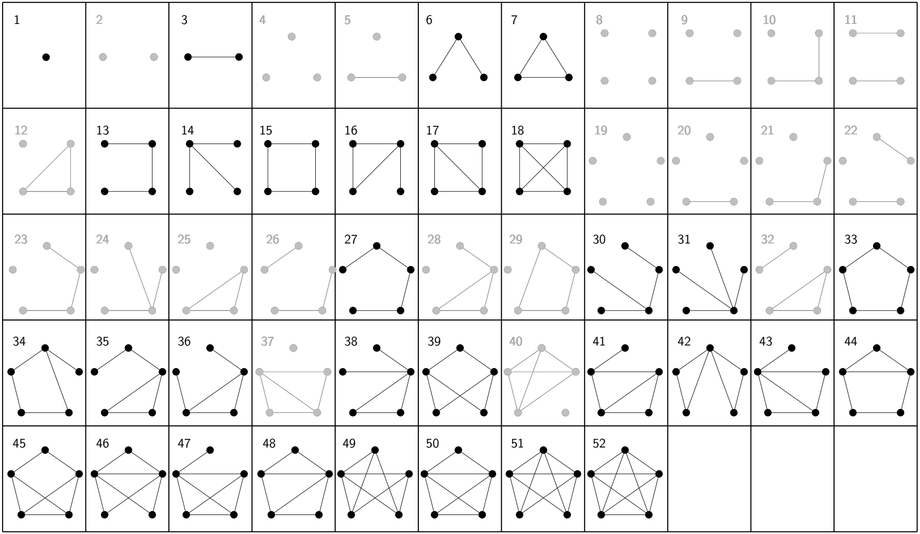

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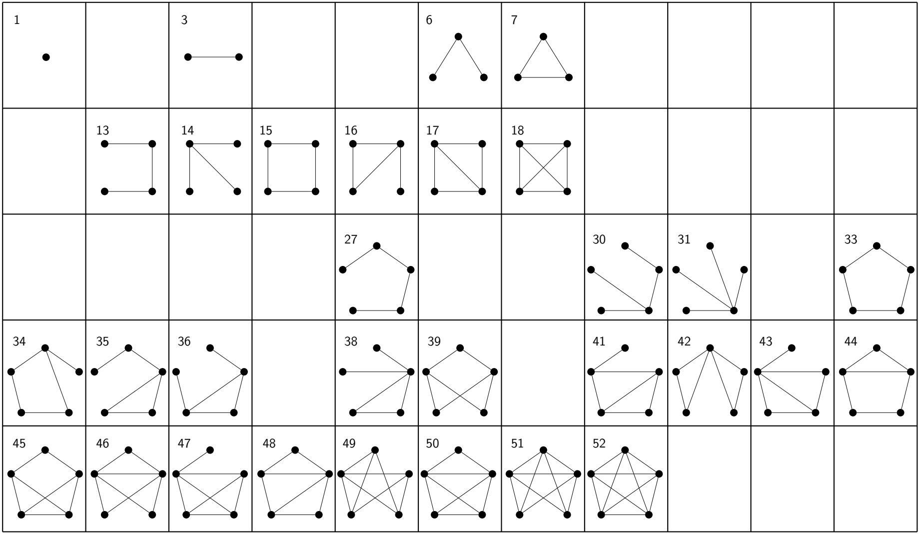

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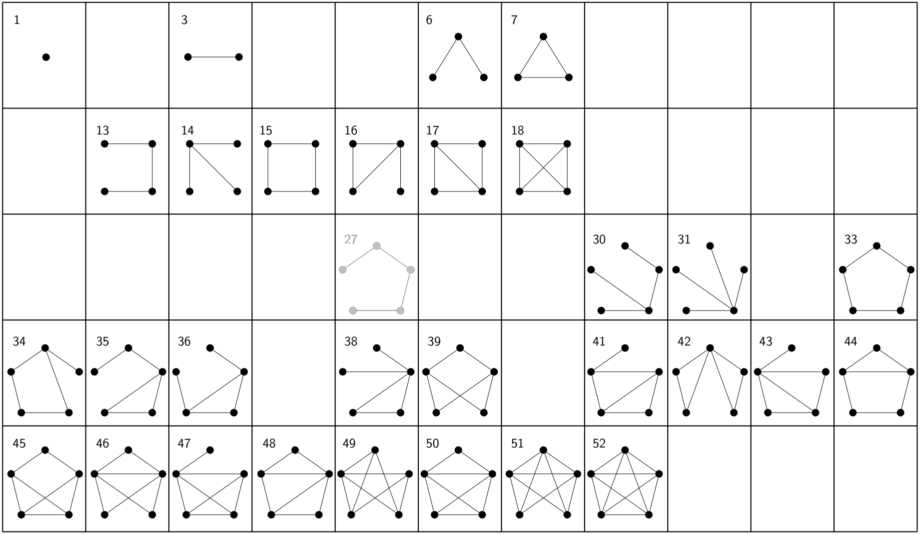

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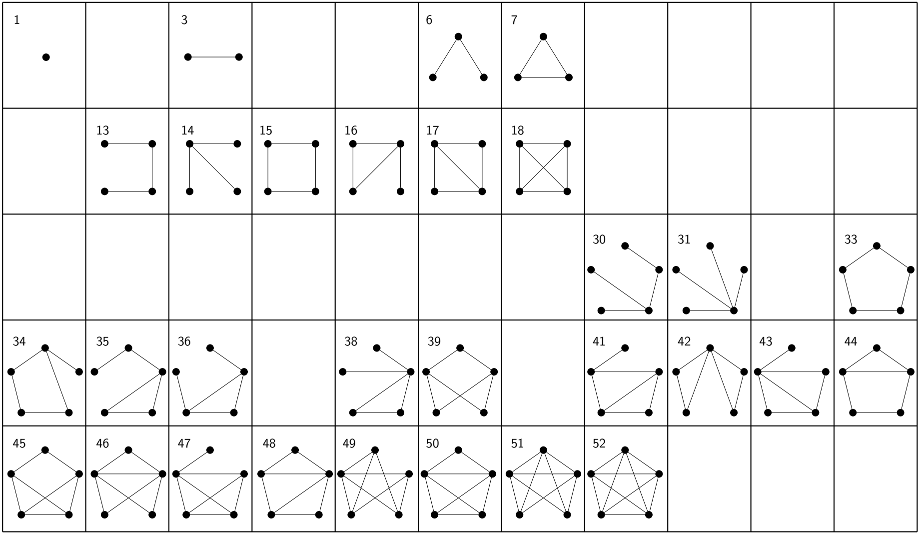

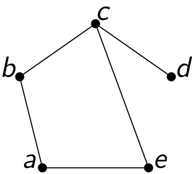
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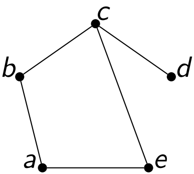

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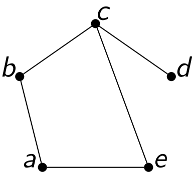

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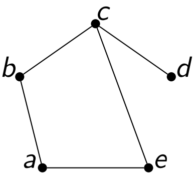
Proof.

Consider $b + e$. This element is annihilated by a and d, but not c. Then $b + e$ can only be either b or e. However, if $b + e = b$, then $e = 0$ an contradiction. Likewise, if $b + e = e$, then $b = 0$, another contradiction.



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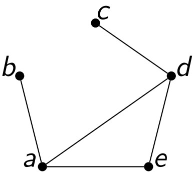
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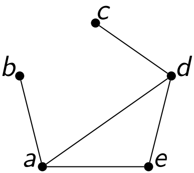

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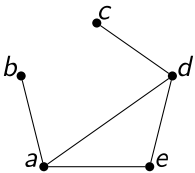

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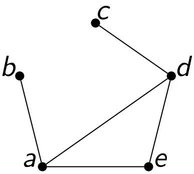

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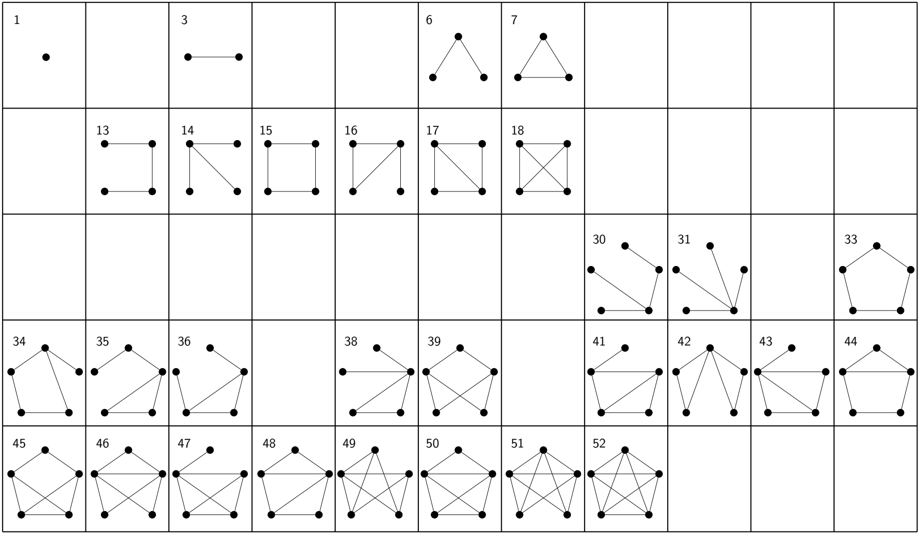

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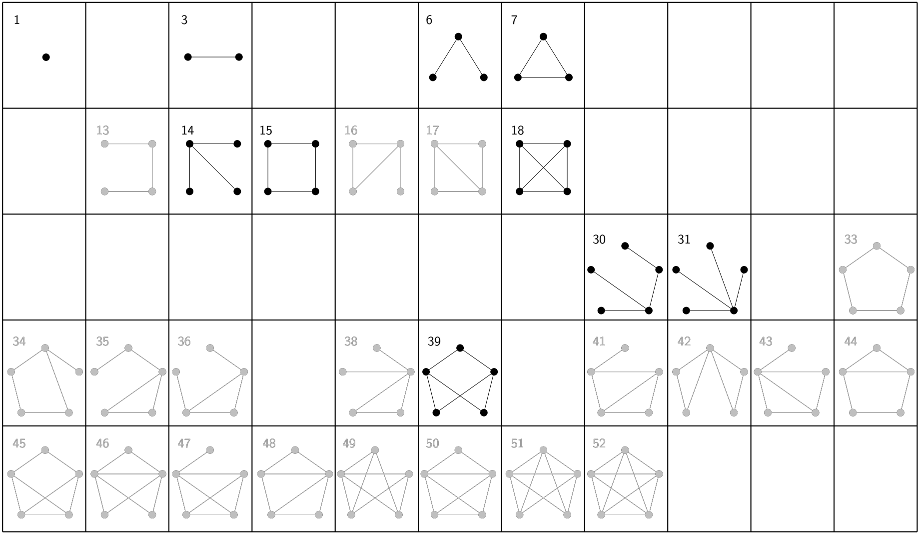

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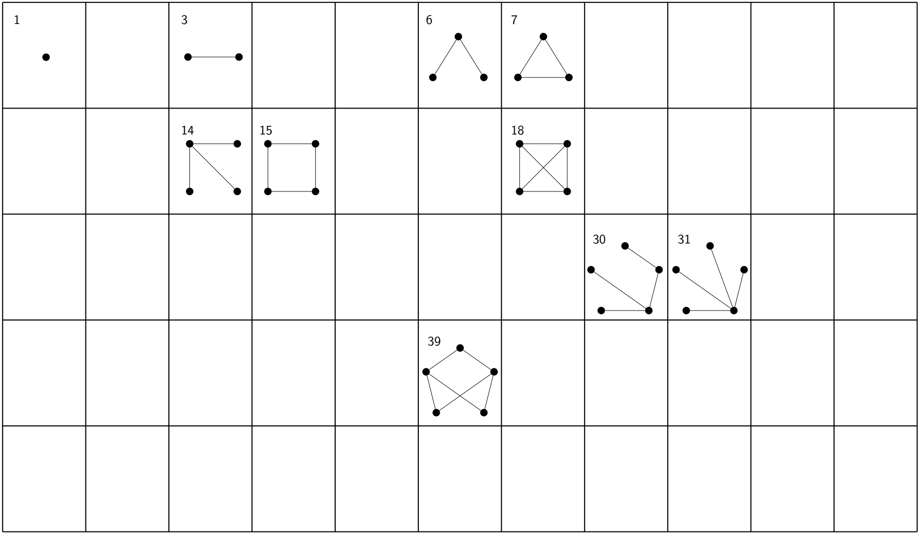
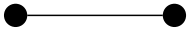
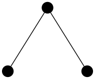
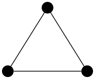
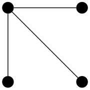
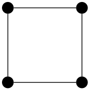
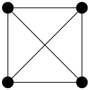
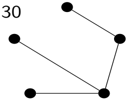
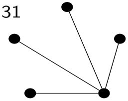
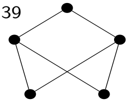


Graphs Realized as AL Graphs

1 	$\mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{<x^2>}$
3 	$\mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_2$
6 	$\mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_3, \frac{\mathbb{Z}_2[x]}{<x^3>}$
7 	$\frac{\mathbb{Z}_2[x,y]}{<x^2,xy,y^2>}$
14 	$\mathbb{Z}_2 \times \mathbb{F}_4$

15 	$\mathbb{Z}_3 \times \mathbb{Z}_3$
18 	$\mathbb{Z}_{25}, \frac{\mathbb{Z}_5[x]}{<x^2>}$
30 	$\mathbb{Z}_2 \times \mathbb{Z}_4$
31 	$\mathbb{Z}_{10}, \mathbb{Z}_2 \times \mathbb{Z}_5$
39 	

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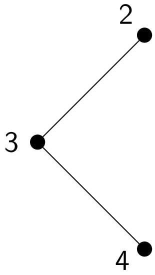

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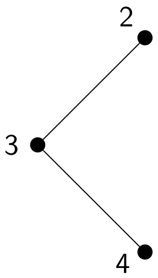

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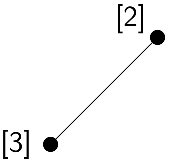

$\Gamma_E(\mathbb{Z}_6)$

Larger Graphs

Mathematicians have discovered the possible [AL] zero-divisor graphs of up to 14 vertices. Past that, there are a significantly larger number of graphs to consider.

Work can be done to find more ways of being able to eliminate more "types" of graphs.