ABEL'S THEOREM

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ABSTRACT. Abel's Theorem is a classical result in the theory of Riemann surfaces. Important in its own right, Abel's Theorem and related ideas generalize to shed light on subjects of current interest in algebraic geometry. In this paper, we will explore some classical material related to Abel's Theorem and give a statement of the theorem. At the end, we very briefly mention some related ideas which are more recent.

1. Preliminary Examples

Consider the indefinite integral

$$\int R(x,y)dx \tag{1.1}$$

where R is a rational function, and x and y are related by a polynomial equation $f(x,y) = 0$. Such an integral is called an *Abelian integral*. For example, the equations $f(x,y) = y^2 - x^2 - ax - b$ and $R(x,y) = 1/y$ lead to the Abelian integral

$$\int \frac{dx}{\sqrt{x^2 + ax + b}} \tag{1.2}$$

Integrals of this form are easily solved in terms of elementary functions; for example

$$\int \frac{dx}{\sqrt{x^2 - c^2}} = ln|x + \sqrt{x^2 - c^2}| + C$$

On the other hand, let $f(x,y) = y^2 - x^3 - ax^2 - bx - c$ and $R(x,y) = 1/y$. These choices lead to the Abelian integral

$$\int \frac{dx}{\sqrt{x^3 + ax^2 + bx + c}} \tag{1.3}$$

This is an example of an *elliptic integral*, which cannot be solved in terms of elementary functions.

In the 1820's, Abel provided some understanding of Abelian integrals. What is today called Abel's Theorem is one important result in the theory that has developed around this subject. Although we will use some terminology that Abel did not have at his disposal, the essence of the theorem is rightfully his.

The equation $f(x, y) = 0$ defines an algebraic plane curve. Since we will be discussing algebraic curves throughout this paper, we begin by clearing up two possible points of confusion about them. The first involves real versus complex dimension. Since we are working over the complex numbers, a curve has complex dimension one but real dimension two. For instance, $\mathbb{C}$ itself has complex dimension one but real dimension two, which is why it is often called the complex plane. We do not use this terminology; for us, a plane has complex dimension two and real dimension four.

The second possible point of confusion is that we add so-called *points at infinity* to make our curves compact. The simplest example of this is adding a single point at infinity to $\mathbb{C}$ to obtain the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Loosely speaking, the curves we are considering look like the surfaces of doughnuts with multiple holes; more precisely, they have the topology of compact orientable surfaces. (For those familiar with projective geometry, when speaking of the curve given by the vanishing of $f(x, y) = y^2 - x^2 - ax - b$ above, we really mean the curve in the projective plane $\mathbb{P}^2$ defined by the vanishing of the homogeneous polynomial $F(X, Y, Z) = Y^2 - X^2 - aXZ - bZ^2$.)

It is an important and nontrivial fact that nonsingular algebraic curves over the complex numbers are essentially the same as Riemann surfaces. Thus, while we will mostly use the language of Riemann surfaces (meaning that we will work complex-analytically) it is worth keeping in mind that the Riemann surfaces we are dealing with can actually be defined algebraically (although not all are *plane* curves). It is possible for the equations in the examples above to define singular curves, but we will assume for simplicity that our curves are nonsingular and hence are Riemann surfaces.

We can think of the Abelian integral given by f and R as a line integral on the Riemann surface (algebraic curve) defined by f. (It is worth a moment to consider how to relate this conceptually to the idea of integrating R over an interval on the real x-line, which is the original context of such integrals.) Note that by line integral we mean an integral along a path (of real dimension 1) on the surface. At this point we leave the path of integration unspecified. An important consideration will be to what extent integration is independent of path in various contexts.

The reason the first Abelian integral above is easy to solve and the second is not is essentially because the Riemann surfaces X_1 and X_2 defined by the vanishing of the polynomials $f_1(x,y) = y^2 - x^2 - ax - b$ and $f_2 = (x,y)y^2 - x^3 - ax^2 - bx - c$ have very different properties. X_1 has the topology of the sphere S^2 (genus 0) while X_2 has the topology of the torus $S^1 \times S^1$ (genus 1). This is an example of the *degree-genus formula* for nonsingular plane curves

$$g(X) = \frac{(d-1)(d-2)}{2}$$

Here g is the topological genus of the curve X and d is the degree of the polynomial equation defining X. It is a fact that the only Riemann surface of genus 0 up to isomorphism is the Riemann sphere. So X_1 is isomorphic to the Riemann sphere, and this allows us to rewrite the first Abelian integral as the integral of a rational function of one variable, which is easily solved. No such simple treatment is possible for the second integral. Moreover, since X_2 is not simply connected, we run into problems with the path of integration.

Abel showed that under certain circumstances, a collection of several integrals over different paths on a Riemann surface are subject to relations; that is, even though a particular integral may be highly complicated, the integrals can sometimes be expressed in terms of each other. To explore this further, we must be more precise and introduce some additional notation.

2. Basic Notions

Let X be a compact Riemann surface of genus g. Topologically, X is just the compact orientable surface of genus g (it looks like the surface of a doughnut with g holes; we also sometimes say it has g handles). The first integral homology group of X, denoted by $H_1(X, \mathbb{Z})$, is a complex vector space of dimension $2g$. It is generated by the homology classes of loops going around each of the holes (g loops) together with the homology classes of loops transverse to these, going around each of the handles (g loops).

Now let ω be a closed C^∞ 1-form on X. (Very briefly for those unfamiliar with the formalism of differential forms, think of ω as something that can be integrated along a path in X. The expression $R(x,y)dx$ in the preceding section is a 1-form. It would be a C^∞ 1-form if R were a C^∞ function. There is an operation called exterior differentiation taking a 1-form ω to a 2-form $d\omega$. We say ω is *closed* if $d\omega = 0$.) If D is a 2-dimension region of X, then by Stokes' theorem:

$$\int_{\partial D} \omega = \int_D d\omega = \int_D 0 = 0$$

Note that the second integral is really a double integral over D, which has real dimension 2 ($d\omega$ is a 2-form, something which can be integrated over a 2-dimensional region). What this tells us is that the integral of a closed 1-form over any boundary chain in X is zero. In other words, the integral of ω around a closed 1-chain c on X depends only on the homology class $[c]$ of c in $H_1(X, \mathbb{Z})$.

A C^∞ 1-form ω on X is called *holomorphic* if it can be written locally in the form $f(z)dz$ for some complex coordinate z and holomorphic function f, where f holomorphic means that $\frac{\partial f}{\partial \bar{z}} = 0$ (complex form of the Cauchy-Riemann equations). For those having some familiarity with differential forms, let us verify that holomorphic 1-forms are closed. Indeed, if ω is holomorphic, then

$$d\omega = d(f(z)dz) = \frac{\partial f}{\partial z} dz \wedge dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz = 0 + 0 = 0$$

A fact which we will not prove is that the holomorphic 1-forms on a compact Riemann surface of genus g form a g-dimensional complex vector space, which we denote $\Omega^1(X)$. Let c be a closed 1-chain on X. Consider the map from $\Omega^1(X)$ to $\mathbb{C}$:

$$\omega \mapsto \int_c \omega \tag{2.1}$$

Since we showed that the integral of a closed form around a boundary is zero, we see that if we integrate ω over $c + \partial D$ for some region D in X, we get the same answer. To reiterate what was stated above about closed forms in general, this means that the map is well defined for homology classes; in other words, it makes sense to write

$$\omega \mapsto \int_{[c]} \omega \tag{2.2}$$

where $[c]$ is the homology class of c, since it does not matter which closed chain in the homology class we integrate over. It is easy to see that the map is linear since

$$\int_{[c]} a\omega_1 + b\omega_2 = a \int_{[c]} \omega_1 + b \int_{[c]} \omega_2$$

where a and b are complex numbers, so we have defined a linear functional from $\Omega^1(X)$ to C.

This special type of linear functional is called a *period.* Since $\Omega^1(X)$ is a g-dimensional complex vector space, the dual space space $\Omega^1(X)^*$ of all linear functionals from $\Omega^1(X)$ to C is also g-dimensional, and the periods form a discrete subgroup Λ of $\Omega^1(X)^*$. In fact, if we think of $\Omega^1(X)^*$ as $\mathbb{C}^g$, then Λ is a *maximal lattice* in $\mathbb{C}^g$ and the quotient group $\mathbb{C}^g/\Lambda$ is a g-dimensional complex torus and therefore compact. It is called the Jacobian of X and denoted $Jac(X)$.

For example, suppose X is a complex torus (genus 1). Then $\Omega^1(X)$ is one-dimensional; any two nonzero holomorphic 1-forms on X are complex multiples of each other. The dual space $\Omega^1(X)^*$ is also one-dimensional; it is isomorphic to $\mathbb{C}$ as a complex vector space. There are two independent periods, corresponding to integration around each of the two homology classes in $H_1(X, \mathbb{Z})$. These two periods, thought of as points in $\mathbb{C}$ via the isomorphism $\Omega^1(X)^* \cong \mathbb{C}$, are linearly independent as vectors in $\mathbb{R}^2 \leftrightarrow \mathbb{C}$, and so the collection of all integer linear combinations of the periods forms a maximal lattice Λ in $\mathbb{C}$. Thus, $\mathbb{C}/\Lambda$ is again a torus. It is a fact that if X is a torus, then $Jac(X) \cong X$. This involves more that just the fact that the two are both topological torii, but we will not go into the details.

A completely trivial example occurs when X is the Riemann Sphere. In this case $\Omega^1(X) = 0$ (there are no nonzero global holomorphic 1-forms!) so $Jac(X)$ is trivial.

Abel's Theorem involves a map into $Jac(X)$. Before stating the theorem, we must discuss the concept of *divisors.* The group of divisors on a compact Riemann surface X is the free abelian group on the points of X. Thus, a divisor on X can be thought of as a finite collection of points on X with multiplicities. We write a divisor as $\sum_{p \in X} n_p p$, where n_p is the multiplicity at p, and where the sum is finite. If this definition seems unnatural, it may be helpful to think of a simple situation in which divisors arise. Consider the meromorphic function $f(z) = \frac{(z-1)^2}{z}$ on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, the simplest example of a Riemann surface. The function f has a pole of order 1 at $z = 0$ and a zero of order 2 at $z = 1$. We also say that it has a pole at ∞ because the absolute value of f becomes arbitrary large as we go away from the origin. We associate to f the divisor $2\{1\} - \{0\} - \{\infty\}$. Note that zeros correspond to positive multiplicities and poles correspond to negative multiplicities. More generally, if a meromorphic function f has zeros of order n_i at points p_i and poles of order m_j at points q_j, then we associate to f a divisor denoted $div(f)$ and defined by:

$$div(f) = \Sigma_i n_i p_i - \Sigma_j n_j q_j$$

The degree $Deg(D)$ of a divisor $D = \sum_{p\in X} n_p p$ is just the sum of the multiplicities:

$$Deg(D) = \sum_{p\in X} n_p \tag{2.3}$$

The degree is an integer since the sum is finite.

The divisors of degree zero form a subgroup $Div_0(X)$ of the group of divisors $Div(X)$ since degree is additive:

$$Deg(D_1 + D_2) = Deg(D_1) + Deg(D_2)$$

A fact which we will not prove is that the divisor of a meromorphic function has degree zero. The Abel-Jacobi map, which is defined in the next section, is a map from $Div_0(X)$ to $Jac(X)$.

3. The Abel-Jacobi Map

Fix a particular point p_0 (called the basepoint) in our Riemann surface X. For any other point $p \in X$ choose a path γ in X from p_0 to p. For a holomorphic 1-form ω on X we take the definite integral

$$\int_{p_0}^{p} \omega$$

along γ and ask in what way this integral depends on the path of integration γ and the basepoint p_0.

If we choose another path from p_0 to p in X and integrate, the difference of the two integrals is the integral of a closed form around a closed path in X. This integral is not zero in general because X is not simply connected in general; a closed path is not generally a boundary. However, a closed path defines a homology class in X, so the difference between the two integrals is the integral of a closed form over a homology class; that is, a period. So while the map

$$\omega \mapsto \int_{p_0}^{p} \omega$$

(leaving the path of integration unspecified) is not well defined as a linear functional $\Omega^1(X) \to \mathbb{C}$, the linear functionals that result from any two particular choices of path differ by a period. We see that choosing a point $p \in X$ (with the basepoint p_0 still fixed) gives a class of linear functionals $\Omega^1(X) \to \mathbb{C}$ (given by integration over various paths from p_0 to p), any two of which differ by a period. Such a class of linear functionals is an element of $Jac(X)$, so we have defined a map $J : X \to Jac(X)$ sending $p \in X$ to the corresponding class of linear functionals. This is almost the Abel-Jacobi map, but not quite. Note that J still depends on the basepoint p_0, since if we choose a different basepoint p_1, two linear functionals defined by integration from p_0 to p and from p_1 to p along two different paths differ by integration over a path that is not even closed.

We can extend J to a map $Div(X) \to Jac(X)$ (which we still call J) by linearity:

$$J(\sum_{p\in X} n_p p) = \sum_{p\in X} n_p J(p)$$

In particular, J restricts to a map $J : Div_0(X) \to Jac(X)$. This restricted map does not depend on the choice of basepoint; if we choose a different basepoint, then all linear functionals we obtain differ by integration around a closed path from those we obtained using the original basepoint. Thus the differences are periods, and the map to the Jacobian is the same for both basepoints. Perhaps the easiest way to see this is to consider a divisor of the form $p - q$ and draw out the paths involved. The general case follows from the fact that any divisor of degree zero can be written as a sum of divisors of the form $p - q$.

The map $J : Div_0(X) \to Jac(X)$ is the Abel-Jacobi map.

4. Statement of Abel's Theorem

In this section we state Abel's Theorem. The proof would require much more work and is not included. Our first example of a divisor was the divisor of a meromorphic function. Although every divisor of degree zero on the Riemann sphere is the divisor of a meromorphic function, this is not true for general Riemann surfaces. An important problem is to answer the question of which divisors on an arbitrary Riemann surface are divisors of meromorphic functions. This problem is called the *Mittag-Leffler problem*. It is important essentially because it involves a passage from *local* to *global* data. The condition for a function to be holomorphic or meromorphic is a local condition; for example a function is holomorphic at a point if it can be expanded in a power series at that point. Thus, the Mittag-Leffler problem is trivially solvable locally: at a particular point $p \in X$ we can always choose a coordinate z which is zero at p. Then locally the divisor np is just the divisor of the local meromorphic function z^n. But does this function extend to a *global* meromorphic function? More generally, if we specify multiplicities n_i at points p_i on X (that is, if we specify a divisor $\Sigma_i n_i p_i$ on X), can we find local meromorphic functions f_i with $div(f_i) = n_i p_i$ locally which patch together to give a global meromorphic function f

on X with $div(f) = \Sigma_i n_i p_i$ globally? Abel's Theorem tells exactly when a divisor of degree zero is the divisor of a global meromorphic function.

Theorem 4.1 (Abel's Theorem)**.** *Let X be a compact Riemann surface and D a divisor of degree zero on X. Then D is the divisor of a global meromorphic function on X if and only if $J(D) = 0$.*

There are various other ways to word the theorem. For example, we can say that a divisor $D = \sum n_p p$ (of degree zero) is the divisor of a meromorphic function on X if and only if for any basepoint $p_0 \in X$ there is a collection of paths λ_p from p_0 to p such that

$$\sum_p \int_{\lambda_p} \omega = 0$$

Obviously if this last statement is true then $J(D) = 0$. Conversely, if $J(D) = 0$, then for any collection of paths λ_p from p_0 to p we choose, the functional $\omega \mapsto \sum_p \int_{\lambda_p} \omega$ is a period, and so $\sum_p \int_{\lambda_p} \omega = \int_{[c]} \omega$ for some homology class $[c]$ represented by a closed path c in X. We form a new path by going around c in the reverse direction and then to one of the points p'. This path together with the other original paths λ_p for $p \neq p'$ form a collection such that the above sum is zero.

Looking back at our original examples involving Abelian integrals, we see that if $\Sigma_i n_i p_i$ is the divisor of a meromorphic function, then Abel's Theorem gives us a relation among the Abelian integrals $\int_{p_0}^{p_i} \omega$.

5. Recent Developments

We now briefly mention some areas of current interest in algebraic geometry that are related to Abel's Theorem.

There is a generalization of the Abel-Jacobi map that is of great interest in modern algebraic geometry. It is the subject of intense research activity and many unanswered questions. To discuss it, we must introduce the concept of algebraic cycles.

As mentioned in the first section, the Riemann surfaces we have been considering are algebraic curves; that is, they are given by the vanishing of polynomial equations. An algebraic variety is a more general object which is defined locally by the vanishing of polynomial equations. We restrict our attention to projective algebraic manifolds, those varieties that can be embedded as closed submanifolds of complex projective n-space for some n. It is an important result (Chow's Theorem) that every closed embedded submanifold of P^n is the zero locus of a finite number of homogeneous polynomials, so the class of manifolds we are considering is actually quite large.

A subvariety of an algebraic variety is a subset which is given locally by the vanishing of polynomials. The nontrivial subvarieties of Riemann surfaces are simply collections of points; for example the subset $\{1\} \cup \{0\}$ of the Riemann sphere is the subvariety defined by the vanishing of the polynomial $z(z-1)$. For general algebraic varieties, there are subvarieties of different dimensions, just as the circle and the sphere can be considered submanifolds of $\mathbb{R}^3$ of different dimensions.

Just as we formed the free abelian group on the set of points of a Riemann surface X to obtain the group of divisors $Div(X)$, we can form the free abelian group on the set of (irreducible) subvarieties of a given codimension k of a general (n-dimensional) algebraic variety V. This group is denoted $Z^k(V)$ and called the *group of codimension-k cycles on* V. Codimension-1 cycles are called divisors (that is, $Z^1(V) = Div(V)$ by definition). For Riemann surfaces, which have complex dimension one, collections of points have codimension one (dimension zero), so this definition agrees with our earlier definition of the group of divisors on a Riemann surface. In general, the groups $Z^k(V)$ are very large and poorly understood. For divisors ($Z^1(V)$) much more is known.

There is a natural way to associate a cohomology class in $H^{2k}(V, \mathbb{Z})$ to a codimension-k cycle. This gives a map

$$cl_k : Z^k(V) \to H^{2k}(V, \mathbb{Z})$$

called the *cycle class map*. (In this section we are following the standard convention of describing everything in terms of *co*dimension and *co*homology. Under certain assumptions, we could describe the same things using dimension and homology.)

Incidentally, the question of whether or not the cycle class map is surjective in some sense is the subject of the famous *Hodge Conjecture*. It is known that the map is not surjective in a strong sense, but it may be surjective in a weaker sense (modulo torsion).

The kernel of the cycle class map consists of the *cycles homologically equivalent to zero* denoted ${Z^k}_{hom}(V)$. In the case of Riemann surfaces, ${Z^k}_{hom}(V)$ is just $Div_0(V)$, the group of divisors of degree zero. This is because a divisor of the form $p - q$ is a boundary (of a 1-chain which is a path from p to q) and any element of $Div_0(V)$ is a sum of such divisors. (This uses the description of the cycle class map in terms of homology alluded to above.)

There is a map from ${Z^k}_{hom}(V)$ into a certain compact complex torus $J^k(X)$ (called the *kth Griffiths Jacobian* of X) which is a generalization of the Abel-Jacobi map we defined above, and which is also called the Abel-Jacobi map. This generalized Abel-Jacobi map exhibits bizarre behavior and is the subject of much current research.

Note that the domain of the Abel-Jacobi map is the kernel of the cycle class map. It is thought that there may be a sequence of maps beginning with the cycle class map and the Abel-Jacobi map, each map defined on the kernel of the preceeding one. Successfully defining and describing these maps would give great insight into the structure of algebraic cycles.

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