Notes on Simplicial Theory I

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January 3, 2013

1 Simplicial Basics.

1.1 The Simplicial Category.

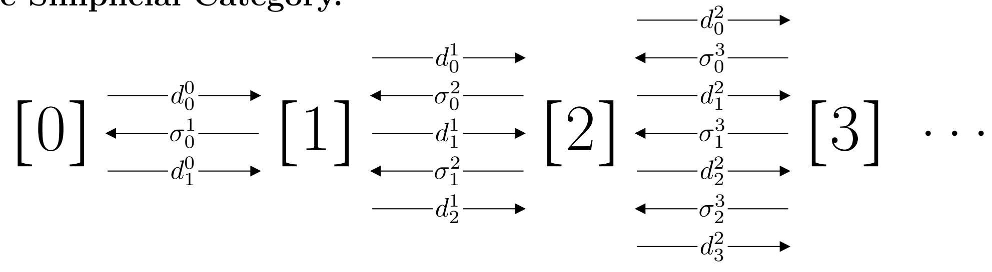
Let Δ be the category whose objects are finite linearly ordered sets $[n] := \{0, 1, ..., n\}$, called **simplices**, and whose morphisms are monotone nondecreasing functions. Δ is called the **simplicial category.** The morphisms in Δ are conveniently described in terms of special morphisms called **face morphisms** (or maps) and **degeneracy morphisms** (or maps). The face morphism $d_i^n : [n] \to [n+1]$ is the unique injective morphism that omits the element i in the image, and the degeneracy morphism $\sigma_i^n : [n] \to [n-1]$ is the unique surjective morphism that has two elements mapping to i. More concretely,

$$d_i^n(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geq i, \end{cases}$$

and

$$\sigma_i^n(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i. \end{cases}$$

The Simplicial Category.



I have chosen to use superscripts n to indicate which simplex $[n]$ is the source of the given face or degeneracy morphism, and subscripts i, j etc. to indicate which face or degeneracy morphism with source $[n]$ is being considered. There are $n+2$ face morphisms with source $[n]$ and n degeneracy morphisms with source $[n]$. Equivalently, there are $n+1$ face morphisms and n degeneracy morphisms between $[n]$ and $[n+1]$. The face and degeneracy morphisms satisfy the following identities:

$$d_j^{n+1} d_i^n = d_i^{n+1} d_{j-1}^n \text{ if } i < j,$$

$$\sigma_j^{n-1} \sigma_i^n = \sigma_i^{n-1} \sigma_{j+1}^n \text{ if } i \leq j,$$

and

$$\sigma_j^{n+1} d_i^n = \begin{cases} d_i^{n-1} \sigma_j^n & \text{if } i < j, \\ \mathrm{Id}_{[n]} & \text{if } i = j \text{ or } i = j+1, \\ d_{i-1}^{n-1} \sigma_j^n & \text{if } i > j+1. \end{cases}$$

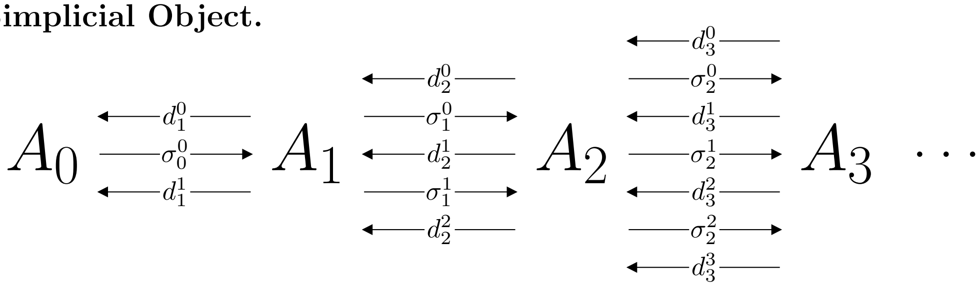
Every morphism $\alpha : [n] \to [m]$ in the simplicial category has a unique factorization $\alpha = \alpha_2 \alpha_1$, where α_1 is an epimorphism consisting of a composition of degeneracy morphisms, and α_2 is a monomorphisms consisting of a composition of face morphisms.

1.2 Simplicial and Cosimplicial Objects.

Let $\mathcal{A}$ be a category. Simplicial and cosimplicial objects in $\mathcal{A}$ are sequences of objects in $\mathcal{A}$ with families of morphisms between them that either reproduce or reverse the abstract structure of the face and degeneracy morphisms in the simplicial category. Unfortunately, the objects that reverse the structure are called "simplicial," while the objects that reproduce the structure are called "cosimplicial." This terminological stumbling block is a historical artifact that is too ingrained to contravene.

Formally, a **simplicial object** in $\mathcal{A}$ is a contravariant functor from Δ to $\mathcal{A}$. More concretely, a simplicial object in $\mathcal{A}$ is a sequence $A_0, A_1, \ldots$ of objects in $\mathcal{A}$, together with **face morphisms** $d_n^i : A_n \to A_{n-1}$ for $0 \leq i \leq n$ and **degeneracy morphisms** $\sigma_n^i : A_n \to A_{n+1}$ for $0 \leq i \leq n$:

A Simplicial Object.



Note that the face and degeneracy morphisms of a simplicial object go in the *opposite directions* from the face and degeneracy morphisms in the simplicial category Δ. This is a confusing artifact of the historical choice to use the term "simplicial object" to denote to a *contravariant* functor from Δ to $\mathcal{A}$ rather than a covariant functor. My notation reflects the difference by using subscripts n to indicate which object A_n in the sequence $A_0, A_1, ...$ is the source of the given face or degeneracy morphism, and superscripts i, j etc. to indicate which face or degeneracy morphism with source A_n is being considered.

The face and degeneracy morphisms must satisfy the following identities:

$$d^i_{n+1} d^j_n = d^{j-1}_{n+1} d^i_n \text{ if } i < j,$$

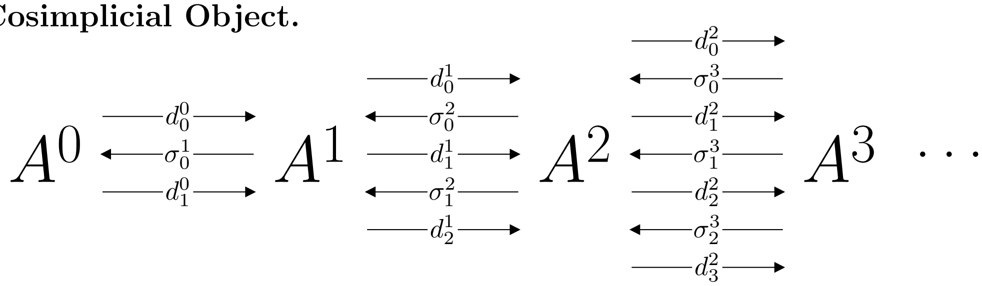
$$\sigma^i_{n-1} \sigma^j_n = \sigma^{j+1}_{n-1} \sigma^i_n \text{ if } i \leq j,$$

and

$$d^i_{n+1} \sigma^j_n = \begin{cases} \sigma^{j-1}_{n-1} d^i_n & \text{if } i < j, \\ \mathrm{Id}_{A_n} & \text{if } i = j \text{ or } i = j+1, \\ \sigma^j_{n-1} d^{i-1}_n & \text{if } i > j+1. \end{cases}$$

A **cosimplicial object** in $\mathcal{A}$ is a covariant functor from Δ to $\mathcal{A}$. More concretely, a simplicial object in $\mathcal{A}$ is a sequence $A^0, A^1, ...$ of objects in $\mathcal{A}$, together with **coface morphisms** $d^n_i : A^n \to A^{n+1}$ and **codegeneracy morphisms** $\sigma^n_i : A^n \to A^{n-1}$:

A Cosimplicial Object.



Note that the coface and codegeneracy morphisms of a cosimplicial object go in the *same directions* as the face and degeneracy morphisms in the simplicial category Δ. My notation reflects this by using superscripts n to indicate which object A^n in the sequence $A^0, A^1, \ldots$ is the source of the given coface or codegeneracy morphism, and subscripts i, j etc. to indicate which coface or codegeneracy morphism with source A^n is being considered.

The coface and codegeneracy morphisms must satisfy the following identities:

$$d_j^{n+1} d_i^n = d_i^{n+1} d_{j-1}^n \text{ if } i < j,$$

$$\sigma_j^{n-1} \sigma_i^n = \sigma_i^{n-1} \sigma_{j+1}^n \text{ if } i \leq j,$$

and

$$\sigma_j^{n+1} d_i^n = \begin{cases} d_i^{n-1} \sigma_{j-1}^n & \text{if } i < j, \\ \mathrm{Id}_{A^n} & \text{if } i = j \text{ or } i = j+1, \\ d_{i-1}^{n-1} \sigma_j^n & \text{if } i > j+1. \end{cases}$$

Note that since the composition $\mathcal{G}\mathcal{F} : \Delta \to \mathcal{B}$ of a contravariant functor $\mathcal{F} : \Delta \to \mathcal{A}$ and a covariant functor $\mathcal{G} : \mathcal{A} \to \mathcal{B}$ is contravariant, a covariant functor from a category $\mathcal{A}$ to a category $\mathcal{B}$ maps simplicial objects of $\mathcal{A}$ to simplicial objects of $\mathcal{B}$. Similarly, a covariant functor from $\mathcal{A}$ to $\mathcal{B}$ maps cosimplicial objects of $\mathcal{A}$ to cosimplicial objects of $\mathcal{B}$, and a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ maps simplicial objects of $\mathcal{A}$ to cosimplicial objects of $\mathcal{B}$ and cosimplicial objects of $\mathcal{A}$ to simplicial objects of $\mathcal{B}$.

1.3 Associated Complexes; Normalized Complexes; Degenerate Complexes.

Associated Chain Complex of a Simplicial Object. Let $A_* = (A_*, d_*^*, \sigma_*^*)$ be a simplicial object in a category $\mathcal{A}$. The **associated chain complex** or **unnormalized chain complex**

$C_*(A_*)$ of A_* is the chain complex whose nth term C_n is A_n and whose nth boundary map $d_n : C_n \to C_{n-1}$ is the alternating sum of the face maps d_n^i:

$$d_n := \sum_{i=0}^{n} (-1)^i d_n^i$$

It is a straightforward calculation to show that $d_{n-1} \circ d_n = 0$ so that C_* is indeed a chain complex.

Normalized Chain Complex; Degenerate Chain Complex. Now suppose that A_* is a simplicial object in an abelian category $\mathcal{A}$. The **normalized chain complex** $N_*(A_*)$ is the subcomplex of $C_*(A_*)$ whose nth term N_n is the intersection of the kernels of the face maps $\bigcap_{i=0}^{n} \ker(d_n^i : A_n \to A_{n-1})$, and whose boundary maps are the restrictions of the boundary maps of $C_*(A_*)$; i.e., the alternating sums of the face maps. The hypothesis that $\mathcal{A}$ is an abelian category is included so that kernels are defined. The **degenerate chain complex** $D_*(A_*)$ is the subcomplex of $C_*(A_*)$ whose nth term D_n is generated by the images of the degeneracy maps $\sigma_{n-1}^i : A_{n-1} \to A_n$, and whose boundary maps are the restrictions of the boundary maps of $C_*(A_*)$. The hypothesis that $\mathcal{A}$ is an abelian category is included so that images are defined. If A_* is a simplicial object in an abelian category, then

$$C_*(A_*) = N_*(A_*) \oplus D_*(A_*).$$

A proof of this result appears in Weibel Chapter 8 Lemma 8.3.7 page 266. Note in particular that because of the simplicial identities

$$d_{n+1}^i \sigma_n^j = \begin{cases} \sigma_{n-1}^{j-1} d_n^i & \text{if } i < j, \\ \mathrm{Id}_{A_n} & \text{if } i = j \text{ or } i = j+1, \\ \sigma_{n-1}^{j} d_n^{i-1} & \text{if } i > j+1, \end{cases}$$

the term $D_{n+1}(A_*)$ maps into $D_n(A_*) \subset C_n(A_*)$. Indeed, $D_{n+1}(A_*)$ is generated by terms of the form $\sigma_n^j(a_n)$ for elements $a_n \in A_n$, and applying the boundary map d_{n+1} yields an alternating sum of terms, all of which are in the image of σ_{n-1}^j or σ_{n-1}^{j-1} except for the two terms for which $i = j$ or $i = j+1$, which have opposite signs and therefore cancel.

It is obvious that $N_{n+1}(A_*)$ maps into $N_n(A_*) \subset C_n(A_*)$ under d_{n+1}, since it maps to 0 under every face map.

Functorial Behavior. If $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is a middle-exact covariant functor between two abelian categories, then $\mathcal{F}$ preserves the splitting of the unnormalized complex; i.e.

$$\mathcal{F}(C_*(A_*)) = \mathcal{F}(N_*(A_*)) \oplus \mathcal{F}(D_*(A_*)).$$

If $\mathcal{F}$ is left-exact (a stronger condition), then $\mathcal{F}$ commutes with C_*, N_*, and D_* in the sense that

$$\mathcal{F}(C_*(A_*)) = C_*(\mathcal{F}(A_*)), \quad \mathcal{F}(N_*(A_*)) = N_*(\mathcal{F}(A_*)), \quad \text{and} \quad \mathcal{F}(D_*(A_*)) = D_*(\mathcal{F}(A_*)).$$

The first equality is always true, because the face and degeneracy maps of $\mathcal{F}(A_*)$ are by definition the images of the face and degeneracy maps of A_* under $\mathcal{F}$. If in addition $\mathcal{F}$ is left-exact, then $\mathcal{F}$ preserves kernels, so $\mathcal{F}(\ker[d_n^i : A_n \to A_{n+1}])$ is equal to $\ker[d_n^i : \mathcal{F}(A_n) \to \mathcal{F}(A_{n+1})]$. Thus, $\mathcal{F}(N_*(A_*)) = N_*(\mathcal{F}(A_*))$, and since the degenerate complex is the direct-sum complement of the normalized complex, $\mathcal{F}(D_*(A_*)) = D_*(\mathcal{F}(A_*))$ as well. Analogous results hold for right-exact covariant functors, left-exact contravariant functors, and right-exact contravariant functors, since either kernels or images are preserved in each case.

Associated Cochain Complex of a Cosimplicial Object. Now, let $A^* = (A^*, d_*^*, \sigma_*^*)$ be a cosimplicial object in a category $\mathcal{A}$. The **associated cochain complex** or **unnormalized cochain complex** $C^*(A^*)$ of A^* is the cochain complex whose nth term C^n is A^n and whose nth coboundary map $d^n : C^n \to C_{n+1}$ is the alternating sum of the coface maps d_i^n:

$$d^n := \sum_{i=0}^{n} (-1)^i d_i^n$$

It is a straightforward calculation to show that $d^{n+1} \circ d^n = 0$ so that C^* is indeed a cochain complex.

Now suppose that A^* is a cosimplicial object in an abelian category $\mathcal{A}$. The **normalized cochain complex** $N^*(A^*)$ is the subcomplex of $C^*(A^*)$ whose nth term N^n is the intersection of the kernels of the coface maps $\bigcap_{i=0}^{n} \ker(d_i^n : A^n \to A^{n+1})$, and whose coboundary maps are the restrictions of the coboundary maps of $C^*(A^*)$; i.e., the alternating sums of the coface maps. The **degenerate cochain complex** $D^*(A^*)$ is the subcomplex of $C^*(A^*)$ whose nth term D^n is generated by the images of the degeneracy maps $\sigma_i^{n+1} : A^{n+1} \to A^n$, and whose coboundary maps are the restrictions of the coboundary maps of $C^*(A^*)$. If A^* is a cosimplicial object in an abelian category, then

$$C^*(A^*) = N^*(A^*) \oplus D^*(A^*).$$

The proof of this result is analogous to the proof for simplicial objects. Similarly, it is true that the coboundary maps d^n map $D^n(A_*)$ into $D^{n+1}(A_*)$ and $N^n(A_*)$ into $N^{n+1}(A_*)$.

Functorial Behavior. As in the simplicial case, middle-exact functors preserve the splitting $C^*(A^*) = N^*(A^*) \oplus D^*(A^*)$, and functors that are either left or right exact commute with taking the unnormalized, degernate, and normalized cochain complexes.

1.4 Simplicial and Cosimplicial Homotopy, Cohomotopy, Homology, and Cohomology.

Let A_* be a simplicial object in a category $\mathcal{A}$. The homology groups $H_n(C_*(A_*))$ of the unnormalized chain complex $C_*(A_*)$ are called the **simplicial homology groups** of A_*. The homology groups $H_n(N_*(A_*))$ of the normalized chain complex $N_*(A_*)$ are called the **simplicial homotopy groups** of A_*, and are denoted by $\pi_n(A_*)$.

Similarly, if A^* is a cosimplicial object in a category $\mathcal{A}$, then the cohomology groups $H^n(C^*(A^*))$ of the unnormalized cochain complex $C^*(A^*)$ are called the **cosimplicial cohomology groups** of A^*. The cohomology groups $H^n(N^*(A^*))$ of the normalized cochain complex $N^*(A^*)$ are called the **cosimplicial cohomotopy groups** of A^*, and are denoted by $\pi^n(A^*)$

If $\mathcal{A}$ is an abelian category, then the simplicial homology groups $H_n(C_*(A_*))$ are isomorphic to the simplicial homotopy groups $\pi_n(A_*)$. Similarly, the cosimplicial cohomology groups $H^n(C^*(A^*))$ are isomorphic to the cosimplicial cohomotopy groups $\pi^n(A^*)$. This follows from the fact that unnormalized complexes are the direct sums of the degenerate and normalized complexes, and the degenerate complexes are acyclic.

1.5 Augmented Simplicial and Cosimplicial Objects.

An bf augmented simplicial object in a category $\mathcal{A}$ is a simplicial object A_* in $\mathcal{A}$ together with a morphism $\epsilon : A_0 \to A$ to a fixed object $A = A_{-1}$ in $\mathcal{A}$ such that $\epsilon d_1^0 = \epsilon d_1^1$. If A has an identity element, then this means that $\epsilon d_1 = 0$, where $d_1 = d_1^0 - d_1^1$ is the first boundary morphism. The morphism ϵ is called the **augmentation morphism.** The meaning of this is that the augmentation allows the associated chain complex $C_*(A_*)$ and normalized chain complex $N_*(A_*)$ to be extended to degree -1 by setting $A_{-1} = A$.

An augmented simplicial object $A_* \to A$ in a category $\mathcal{A}$ is called **aspherical** if the simplicial homotopy groups $\pi_n(A_*)$ vanish for $n \geq 1$, and if the augmentation map ϵ is an isomorphism between $\pi_0(A_*)$ and A_{-1}. In an abelian category, this is equivalent to the assertion that the associated augmented chain complex and the augmented normalized chain complex are exact, meaning that $C_*(A_*)$ and $N_*(A_*)$ are resolutions of $A = A_{-1}$. In this context, $C_*(A_*)$ is sometimes called a **simplicial resolution.**

An augmented simplicial object $A_* \to A$ with augmentation map ϵ is called **right contractible** if there are morphisms $f_n : A_n \to A_{n+1}$ for all n (including $f_{-1} : A_{-1} \to A_0$) such that $\epsilon f_{-1} = \mathrm{Id}$, $d_{n+1}^{n+1} f_n = \mathrm{Id}$ for $n \geq 0$, $d_1^0 f_0 = f_{-1}\epsilon$, and $d_{n+1}^i f_n = f_{n-1} d_n^i$ for all $0 \leq i \leq n$. It is called **left contractible** if $\epsilon f_{-1} = \mathrm{Id}$, $d_{n+1}^0 f_n = \mathrm{Id}$ for $n \geq 0$, $d_1^1 f_0 = f_{-1}\epsilon$, and $d_{n+1}^i f_n = f_{n-1} d_n^{i-1}$ for all $0 \leq i \leq n$. If $A_* \to A$ is either left or right contractible, it is called **contractible**. If $A_* \to A$ is a contractible augmented simplicial object in a category $\mathcal{A}$, then $A_* \to A$ is aspherical, and the associated augmented chain complex and the augmented normalized chain complex are split exact.

1.6 Dold-Kan Correspondence.

The Dold-Kan correspondence says that any chain complex concetrated in nonnegative degrees in an abelian category may be represented by a simplicial object, and every cochain complex concetrated in nonnegative degrees in an abelian category may be represented by a cosimplicial object. More concretely, the functor that assigns to a simplicial object A_* in an abelian category $\mathcal{A}$ its corresponding normalized complex $N_*(A_*)$ is an equivalence between the category of simplicial objects in $\mathcal{A}$ and the category of chain complexes C_* in $\mathcal{A}$ with $C_n = 0$ for $n < 0$, and the functor that assigns to a cosimplicial object A^* in an abelian category $\mathcal{A}$ its corresponding normalized complex $N^*(A^*)$ is an equivalence between the category of cosimplicial objects in $\mathcal{A}$ and the category of cochain complexes C^* in $\mathcal{A}$ with $C^n = 0$ for $n < 0$.

2 Cotriple Theory.

The following brief buildup to cotriple cohomology draws largely from [Weibel] Chapter 8 sections 8.6 and 8.7. The reader may find it helpful to skip back and forth between the abstract definitions here in section **??** and the more concrete constructions in sections **??** and **??**, which show how the abstract theory is actually applied in the present context.

2.1 Triples and Cotriples.

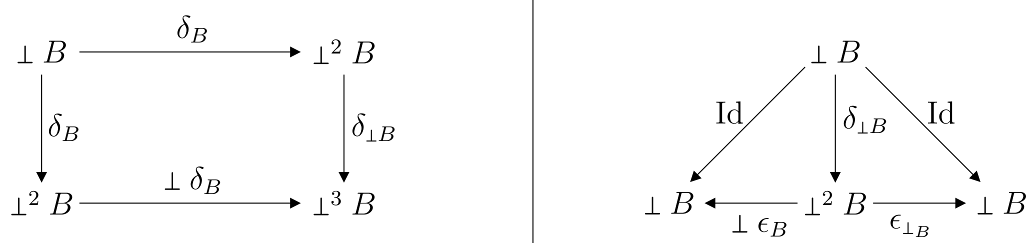
A **triple** $(\top, \eta, \mu)$ on a category $\mathcal{A}$ is a functor $\top : \mathcal{A} \to \mathcal{A}$ together with natural transformations $\eta : \mathrm{Id}_{\mathcal{A}} \to \top$ and $\mu : \top^2 \to \top$ such that the following diagrams commute for every object in $\mathcal{A}$:

Triple Conditions.

$$\begin{array}{ccc} \top^3 A & \xrightarrow{\top\mu} & \top^2 A \\ \downarrow{\scriptstyle \mu_{\top A}} & & \downarrow{\scriptstyle \mu_A} \\ \top^2 A & \xrightarrow{\mu_A} & \top A \end{array} \qquad \begin{array}{ccccc} \top A & \xrightarrow{\top\eta_A} & \top^2 A & \xleftarrow{\eta_{\top A}} & \top A \\ & \searrow{\scriptstyle \mathrm{Id}} & \downarrow{\scriptstyle \mu_{\top A}} & \swarrow{\scriptstyle \mathrm{Id}} & \\ & & \top A & & \end{array}$$

A **cotriple** $(\perp, \epsilon, \delta)$ on a category $\mathcal{B}$ is a functor $\perp : \mathcal{B} \to \mathcal{B}$ together with natural transformations $\epsilon : \perp \to \mathrm{Id}_{\mathcal{B}}$ and $\delta : \perp \to \perp^2$ such that the following diagrams commute for every object in $\mathcal{B}$:

Cotriple Conditions.



2.2 Cotriple of a Pair of Adjoint Functors.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are categories, and suppose $\mathcal{L} : \mathcal{A} \to \mathcal{B}$ is a functor with a **right adjoint** $\mathcal{R} : \mathcal{B} \to \mathcal{A}$. This means that there is a set bijection

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{L}A, B) \cong \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{R}B).$$

The pair of functors $(\mathcal{L}, \mathcal{R})$ is called an **adjoint pair.** $\mathcal{R}$ is called right-adjoint to $\mathcal{L}$, and $\mathcal{L}$ is called left-adjoint to $\mathcal{R}$, because $\mathcal{R}$ appears in the second argument of Hom, while $\mathcal{L}$ appears in the first argument. There is also a "naturality condition" that a pair of functors must satisfy to be called an adjoint pair. This condition can be expressed as the commutativity of a certain diagram; see the appendix of [Weibel], section 6, page 430.

Every adjoint pair $(\mathcal{L}, \mathcal{R})$ of functors determines two natural transformations $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow \mathcal{R}\mathcal{L}$ and $\epsilon : \mathcal{L}\mathcal{R} \Rightarrow \mathrm{Id}_{\mathcal{B}}$ called the **unit** and the **counit of the adjunction**, respectively. The unit η assigns to each object $A \in \mathcal{A}$ the morphism $\eta_A : A \to \mathcal{R}\mathcal{L}A$ in $\mathrm{Hom}_{\mathcal{A}}(A, \mathcal{R}\mathcal{L}A)$ which is the bijective image of the identity morphism $\mathrm{Id}_{\mathcal{L}A} : \mathcal{L}A \to \mathcal{L}A$ under the set bijection $\mathrm{Hom}_{\mathcal{A}}(\mathcal{L}A, \mathcal{L}A) \cong \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{R}\mathcal{L}A)$. The counit ϵ assigns to each object $B \in \mathcal{B}$ the morphism $\epsilon_B : \mathcal{L}\mathcal{R}B \to B$ in $\mathrm{Hom}_{\mathcal{B}}(\mathcal{L}\mathcal{R}B, B)$ which is the bijective image of the identity morphism $\mathrm{Id}_{\mathcal{R}B} : \mathcal{R}B \to \mathcal{R}B$ under the set bijection $\mathrm{Hom}_{\mathcal{B}}(\mathcal{L}\mathcal{R}B, B) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{R}B, \mathcal{R}B)$.

For every adjoint pair $(\mathcal{L}, \mathcal{R})$ there exists a cotriple $(\perp, \epsilon, \delta)$ on the category $\mathcal{B}$, where $\perp$ is the composite functor $\mathcal{L}\mathcal{R} : \mathcal{B} \to \mathcal{B}$, ϵ is the counit of the adjunction, and $\delta : \perp \Rightarrow \perp^2$ is the natural transformation assigning to each $B \in \mathcal{B}$ the morphism $\mathcal{L}\eta_{\mathcal{R}B} : \perp B \to \perp^2 B$.

2.3 Simplicial Object of a Cotriple.

Let $(\perp, \epsilon, \delta)$ be a cotriple on a category $\mathcal{B}$. For any object $B = B_{-1} \in \mathcal{B}$, define a simplicial object B_* in $\mathcal{B}$ as follows. Define $B_n := \perp^{n+1} B$. In particular, $B_0 = \perp B$. Define face and degeneracy maps

$$d_n^i = \perp^i \epsilon \perp^{n-1} : B_n = \perp^{n+1} B \longrightarrow \perp^n B = B_{n-1}$$

$$\sigma_n^i = \perp^i \delta \perp^{n-i} : B_n = \perp^{n+1} B \longrightarrow \perp^{n+2} B = B_{n+1}$$

With these definitions, the sequence $B_* \to B$ is an augmented simplicial object in $\mathcal{B}$, where the map $B_0 = \perp B \to B = B_{-1}$ is ϵ.

2.4 Projective Objects with Respect to Cotriples.

Let $(\perp, \epsilon, \delta)$ be a cotriple on a category $\mathcal{B}$. An object $B \in \mathcal{B}$ is called $\perp$-projective if the map $\epsilon_B : \perp B \to B$ has a section $f : B \to \perp B$; that is, a morphism f such that $\epsilon_B f = \mathrm{Id}_B$. For example, if $\perp = \mathcal{L}\mathcal{R}$ for a pair of functors $\mathcal{L} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ with $\mathcal{L}$ left-adjoint to $\mathcal{R}$, then every object of the form $\mathcal{L}A$ for some $A \in \mathcal{A}$ is $\perp$-projective in $\mathcal{B}$ because the morphism $\mathcal{L}(\eta_A) : \mathcal{L}A \to \mathcal{L}\mathcal{R}\mathcal{L}A = \perp \mathcal{L}A$ is such a section.

2.5 Canonical Resolutions for Cotriples on Abelian Categories.

Let $(\perp, \epsilon, \delta)$ be a cotriple on an abelian category $\mathcal{B}$. If $B \in \mathcal{B}$ is any $\perp$-projective object, then the augmented simplicial object $B_* \to B$ is aspherical, and the associated augmented chain complex $C_*(B_*) \to B$ is exact. Since $\mathcal{B}$ is abelian, the conditions that $C_*(B_*)$ is aspherical and that the associated augmented chain complex is exact are equivalent by the Dold-Kan correspondence. The associated augmented chain complex and the normalized complex are then both resolutions of B, called the **canonical resolutions.**

2.6 Cotriple Cohomology.

Let $(\perp, \epsilon, \delta)$ be a cotriple on a category $\mathcal{B}$, and let $\mathcal{F}$ be a contravariant functor from $\mathcal{B}$ to an abelian category $\mathcal{A}$. Let B be an object in $\mathcal{B}$. In this case, $\mathcal{F}B_*$ is a cosimplicial object in $\mathcal{A}$. The **cotriple cohomology** $H^n(B, \mathcal{F})$ of B with coefficients in F is by definition the cosimplicial cohomotopy $\pi_*(\mathcal{F}B_*)$, which is by definition the cohomology $H^*(N^*(\mathcal{F}B_*))$ of the corresponding normalized complex.

3 Ext Functors as Cotriple Cohomology Functors.

Let R be a ring, and let B, B' be left R-modules. The functors $\mathrm{Ext}^*_R(-, B')$, which are the the right derived functors of the left-exact functor $\mathrm{Hom}_R(-, B')$, may be viewed as cotriple cohomology functors in the following way. Let $\mathcal{A}$ be the category of sets, and let $\mathcal{B}$ be the category of R-modules. Let $\mathcal{L} : \mathcal{A} \to \mathcal{B}$ be the functor assigning to each set A the free R-module over A, and let $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ be forgetful functor. Then $(\mathcal{L}, \mathcal{R})$ is an adjoint pair of functors, with $\mathcal{R}$ right-adjoint to $\mathcal{L}$. To see this, define maps

$$\phi : \mathrm{Hom}_{\mathcal{B}}(\mathcal{L}A, B) \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{R}B) \qquad \text{and} \qquad \psi : \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{R}B) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{L}A, B)$$

as follows. For any set A, denote a generic element of the free module $\mathcal{L}A$ over A by $\sum_i r_i a_i$, where $r_i \in R$, $a_i \in A$, and the sum is finite. Then for any R-module map $f : \mathcal{L}A \to B$,

$$f\Big(\sum_i r_i a_i\Big) = \sum_i r_i \cdot f(1_R a_i)$$

where $r_i \cdot f(1_R a_i)$ denotes the R-action on B. Define $\phi(f) : A \to \mathcal{R}B$ to be the set map sending a to $f(1_R a)$. This construction is obviously reversible: if $g : A \to \mathcal{R}B$ is a set map, then define $\psi(g) : \mathcal{L}A \to B$ to be the R-module map sending $\sum_i r_i a_i$ to $\sum_i r_i \cdot g(a_i)$, where $r_i \cdot g(a_i)$ denotes the R-action on B.

Now define the **free module cotriple** $(\perp, \epsilon, \delta)$ to be the cotriple of the adjoint pair $(\mathcal{L}, \mathcal{R})$ on the category $\mathcal{B}$ of R-modules. The functor $\perp = \mathcal{L}\mathcal{R}$ sends an R-module B to the free R-module based on the underlying set of B. The unit η is the family of set maps that send each set A into the underlying set of the free R-module over A. The counit ϵ is the family of R-module maps that send the free R-module based on the underlying set of B onto B. The natural transformation δ, whose value on an R-module B is the map $\delta_B = \mathcal{L}\eta_{\mathcal{R}B}$, sends $\perp B$ to $\perp^2 B$. For any R-module B, define B_* to be the corresponding simplicial object of the cotriple, whose nth term B_n is $\perp^{n+1} B$. Then the associated chain complex $C_*(B_*)$ is a free resolution of B. The nth cotriple cohomology group $H^n(B, \mathrm{Hom}_R(-, B'))$ of B with coefficients in $\mathrm{Hom}_R(-, B')$ is by definition the cohomotopy $\pi^n \mathrm{Hom}_R(B_*, B')$ of the cosimplicial object $\mathrm{Hom}_R(B_*, B')$, which is by definition the cohomology $H^n(N^*(\mathrm{Hom}_R(B_*, B')))$ of the corresponding normalized complex $N^*(\mathrm{Hom}_R(B_*, B'))$. Since Hom is a functor into an abelian category, this is the same as the cohomology of the unnormalized complex $C^*(\mathrm{Hom}_R(B_*, B'))$ whose nth term is $\mathrm{Hom}_R(B_n, B')$, which is the result of applying the functor $\mathrm{Hom}_R(-, B')$ to the free resolution $C_*(B_*)$ of B. Thus,

$$H^n(B, \mathrm{Hom}_R(-, B')) = \mathrm{Ext}^n_R(B, B').$$

4 Bar Resolutions of Modules and Algebras in the Unital Case.

4.1 Left Bar Resolution of a Module.

Let $\gamma : k \to R$ be a homomorphism of unital rings. Make the following definitions.

- $\mathcal{A}$ is the category of left k-modules.
- $\mathcal{B}$ is the category of left R-modules.

- $\mathcal{L} : \mathcal{A} \to \mathcal{B}$ is the base-change functor $A \mapsto R \otimes_k A$.
- $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ is the forgetful functor.

Then $(\mathcal{L}, \mathcal{R})$ is an adjoint pair of functors via the following maps:

- $\phi : \mathrm{Hom}_R(R \otimes A, B) \to \mathrm{Hom}_k(A, B)$ is the map sending each R-module map $f : R \otimes A \to B$ to the k-module map $\phi(f)$ sending $a \in A$ to $f(1 \otimes a)$.
- $\psi : \mathrm{Hom}_k(A, B) \to \mathrm{Hom}_R(R \otimes A, B)$ is the map sending each k-module map $g : A \to B$ to the R-module map $\psi(g) : R \otimes A \to B$ sending $r \otimes a$ to $r \cdot g(a)$.

Note that the definition of ϕ requires that R be unital.

To see that $(\mathcal{L}, \mathcal{R})$ is an adjoint pair, first check that ϕ and ψ are inverse set maps. Indeed,

$$\psi\phi(f)(r \otimes a) = r \cdot \phi(f)(a) = r \cdot f(1 \otimes a) = f(r \otimes a)$$

since f is an R-module map, and

$$\phi\psi(g)(a) = \psi(g)(1 \otimes a) = 1 \cdot g(a) = g(a).$$

Next, check the naturality condition, which is the commutativity of the following diagram, where $h : A \to A'$ is a k-module map and $\ell : B \to B'$ is an R-module map:

Naturality Conditions for the Adjoint Pair $(\mathcal{L}, \mathcal{R}) = (R \otimes_k -, \mathcal{F})$.

$$\begin{array}{ccccc}
\mathrm{Hom}_R(R \otimes A', B) & \xrightarrow{(\mathrm{Id}_R \otimes h)^*} & \mathrm{Hom}_R(R \otimes A, B) & \xrightarrow{\ell_*} & \mathrm{Hom}_R(R \otimes A, B') \\
\downarrow \phi & & \downarrow \phi & & \downarrow \phi \\
\mathrm{Hom}_k(A', B) & \xrightarrow{h^*} & \mathrm{Hom}_k(A, B) & \xrightarrow{\ell_*} & \mathrm{Hom}_k(A, B')
\end{array}$$

For the left square, suppose $f' \in \mathrm{Hom}_R(R \otimes A', B)$. Then for any $a \in A$,

$$\phi \circ (\mathrm{Id}_R \otimes h)^*(f')(a) = (\mathrm{Id}_R \otimes h)^*(f')(1 \otimes a) = f'((\mathrm{Id}_R \otimes h)(1 \otimes a)) = f'(1 \otimes h(a)) = \phi(f')(h(a))$$

$$= h^* \circ \phi(f)(a)$$

For the right square, suppose $g \in \mathrm{Hom}_R(R \otimes A, B)$. Then for any $a \in A$,

$$\phi \circ \ell^*(g)(a) = \ell_*(g)(a \otimes 1) = \ell_* \circ \phi(g)(a \otimes 1).$$

$\mathrm{Hom}_R(R \otimes A, B)$ is a right k-module under the action $(f \cdot \alpha)(r \otimes a) = f(r \otimes \alpha \cdot a)$, and $\mathrm{Hom}_k(A, B)$ is a right k-module under the action $(g \cdot \alpha)(a) = g(\alpha \cdot a)$.

The maps ϕ and ψ are isomorphisms of right k-modules:

$$\phi(f \cdot \alpha)(a) = (f \cdot \alpha)(1 \otimes a) = f(1 \otimes \alpha \cdot a) = \phi(f)(\alpha \cdot a) = (\phi(f) \cdot \alpha)(a),$$

and

$$\psi(g \cdot \alpha)(r \otimes a) = r \cdot (g \cdot \alpha)(a) = r \cdot g(\alpha \cdot a) = \psi(g)(r \otimes \alpha a) = (\psi(g) \cdot \alpha)(r \otimes a).$$

Now define the cotriple $(\perp, \epsilon, \delta)$ of the map $\gamma : k \to R$ to be the cotriple of the adjoint pair $(\mathcal{L}, \mathcal{R})$ on the category $\mathcal{B}$ of R-modules. The functor $\perp = \mathcal{L}\mathcal{R}$ sends an R-module B to the R-module $R \otimes_k B$.

With this in mind,

- The unit η is the family of k-module maps η_A that send each k-module A into the underlying k-module of the R-module $R \otimes A$ via $a \mapsto 1 \otimes a$. To see this, recall that for any element $f \in \mathrm{Hom}_R(R \otimes A, B)$, η_A is by definition the map $A \to R \otimes A$ such that $\phi(f) : A \to B$ is equal to $f \circ \eta_A$.
- The counit ϵ is the family of R-module maps ϵ_B that send the R-module $R \otimes B$ onto B via $r \otimes b \to r \cdot b$. To see this, recall that for any element $g \in \mathrm{Hom}_k(A, B)$, ϵ_B is by definition the map $R \otimes B \to B$ such that $\psi(g) : R \otimes A \to B$ is equal to $\epsilon_B \circ (\mathrm{Id}_R \otimes g)$.
- The natural transformation δ is the family of R-module maps sending $\perp B$ to $\perp^2 B$ via $r \otimes b \mapsto r \otimes 1 \otimes b$. This is obvious from the definition $\delta_B = \mathcal{L}\eta_{\mathcal{R}B}$.
- The natural transformation μ is the family of k-module maps sending $\top^2 A$ to $\top A$ via $r \otimes r' \otimes a \mapsto rr' \otimes a$. This is obvious from the definition $\mu_A = \mathcal{R}\epsilon_{\mathcal{L}A}$. Recall that $\top$ is the triple functor $\mathcal{R}\mathcal{L}$. The functor $\top$ and the "multiplication" μ are not crucial to this section.

Now for any R-module B, define B_* to be the corresponding simplicial object of the cotriple. The nth term B_n of B_* is $\perp^{n+1} B = R^{\otimes(n+1)} \otimes B$. The associated chain complex $C_*(B_*)$ is an aspherical resolution of B called the **left bar resolution** of B. The term "left" refers to the fact that B and B' are left R-modules. The term "bar" refers to an arbitrary historical artifact of notation.

It is useful to derive a more explicit description of the face maps, boundary maps, and degeneracy maps in the left bar resolution. Begin with the face maps $d_n^i : B_n \to B_{n-1}$. By definition, these are

$$d_n^i = \perp^i \epsilon \perp^{n-i} : B_n = \perp^{n+1} B \longrightarrow \perp^n B = B_{n-1}.$$

First consider d_n^0, which is given by $\epsilon \perp^n$. This reduces to the map $\epsilon_{B_{n-1}} : B_n \to B_{n-1}$ given by the natural transformation $\epsilon : \perp \to \mathrm{Id}$. This map sends

$$r_0 \otimes \ldots \otimes r_n \otimes b = r_0 \otimes (r_1 \otimes \ldots \otimes r_n \otimes b) \qquad \text{to} \qquad r_0 \cdot (r_1 \otimes \ldots \otimes r_n \otimes b) = r_0 r_1 \otimes \ldots \otimes r_n \otimes b$$

Next, consider d_n^i for $0 < i \leq n$, which is given by $\perp^i \epsilon \perp^{n-i}$, where the right-most exponent is at least one. First apply $\epsilon_{B_{n-i}}$ to the identity $B_{n-i} \to B_{n-i}$ to obtain the map sending $r_i \otimes (r_{i+1} \otimes \ldots \otimes r_n \otimes b)$ to $r_i r_{i+1} \otimes \ldots \otimes r_n \otimes b$. Next, apply $\perp^i$ to obtain the map sending

$$r_0 \otimes \ldots \otimes r_n \otimes b \qquad \text{to} \qquad r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_n \otimes b.$$

Finally, consider d_n^n, which is given by $\perp^n \epsilon$. First apply ϵ_B to the identity $B \to B$ to obtain the map sending $r_n \otimes b$ to $r_n \cdot b$. Next, apply $\perp^n$ to obtain the map sending

$$r_0 \otimes \ldots \otimes r_n \otimes b \qquad \text{to} \qquad r_0 \otimes \ldots \otimes r_n \cdot b.$$

Combining these formulas together with alternating signs, the boundary map $d_n : B_n \to B_{n-1}$ in the left bar resolution is given by

Boundary maps for the Left Bar Resolution of a left R-module B.

$$d_n(r_0 \otimes \ldots \otimes r_n \otimes b) = \sum_{i=0}^{n-1} (-1)^i (r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_n \otimes b) + (-1)^n (r_0 \otimes \ldots \otimes r_n \cdot b).$$

4.2 Normalized Left Bar Resolution of a Module.

Now consider the degeneracy maps $\sigma_n^i : B_n \to B_{n+1}$. By definition, these are

$$\sigma_n^i = \perp^i \delta \perp^{n-i} : B_n = \perp^{n+1} B \longrightarrow \perp^{n+2} B = B_{n+1}.$$

First consider σ_n^0, which is given by $\delta \perp^n$. This reduces to the map $\delta_{B_{n-1}} : B_n \to B_{n+1}$ given by the natural transformation $\delta : \perp \to \perp^2$. This map sends

$$r_0 \otimes \ldots \otimes r_n \otimes b \qquad \text{to} \qquad r_0 \otimes 1 \otimes \ldots \otimes r_n \otimes b$$

Similarly, σ_n^i sends

$$r_0 \otimes \ldots \otimes r_n \otimes b \qquad \text{to} \qquad r_0 \otimes \ldots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \ldots \otimes r_n \otimes b$$

for $0 < i < n-1$. In particular, the last degeneracy map σ_n^n inserts a 1 between r_n and b.

The **degenerate left bar resolution** $D_*(B_*)$ is the complex generated by the images of the degeneracy maps. Its nth term $D_n(B_*)$ is generated by elements of the form $r_0 \otimes r_1 \otimes ... \otimes r_n \otimes b$, where at least one of the factors $r_1, ..., r_n$ in the tensor product is equal to 1. Note that since elements of k move across the tensor products, any tensor product for which one of these factors is an element of k is an element of the degenerate complex.

The **normalized left bar resolution** $N_*(B_*)$ is the quotient $C_*(B_*)/D_*(B_*)$. Its nth term $N_n(B_*)$ is $R \otimes \bar{R}^{\otimes n} \otimes B$, generated by elements of the form $r_0 \otimes \bar{r}_1 \otimes ... \otimes \bar{r}_n \otimes b$, where $\bar{r}_i$ is the class of $r_i \in R$ in the cokernel $\bar{R}$ of the map $k \to R$. The boundary maps in the normalized left bar resolution are induced by the boundary maps in the unnormalized left bar resolution:

Boundary maps for the Normalized Left Bar Resolution of a left R-module B.

$$d_n(r_0 \otimes \bar{r_1} \otimes ... \otimes \bar{r}_n \otimes b) = (r_0 r_1 \otimes \bar{r}_2 \otimes ... \otimes \bar{r}_n \otimes b) + \sum_{i=1}^{n-1} (-1)^i (r_0 \otimes \bar{r_1} \otimes ... \otimes \bar{r}_i \bar{r}_{i+1} \otimes ... \otimes \bar{r}_n \otimes b)$$
$$+(-1)^n (r_0 \otimes \bar{r_1} \otimes ... \otimes \bar{r}_{n-1} \otimes r_n \cdot b).$$

Note that in the above formula, the overbars on r_1 and r_n have disappeared in the first and last terms, and it is not *a priori* obvious that the formula is well-defined. Although the boundary maps d_n preserve the degenerate complex by general simplicial theory, it is worthwhile to see explicitly what happens when an element $\alpha \in k$ appears in the place of r_1 or r_n in the case of the left bar resolution. Consider $r_1 = \alpha$:

$$d_n(r_0 \otimes \alpha \otimes r_2 \otimes ... \otimes r_n \otimes b) = (r_0 \alpha \otimes r_2 \otimes ... \otimes r_n \otimes b) - (r_0 \otimes \alpha r_2 \otimes ... \otimes r_n \otimes b)$$

$$+ \sum_{i=2}^{n-1} (-1)^i (r_0 \otimes \alpha \otimes r_2 \otimes ... \otimes r_i r_{i+1} \otimes ... \otimes r_n \otimes b) + (-1)^n (r_0 \otimes \alpha \otimes r_2 \otimes ... \otimes r_n \cdot b).$$

The first two terms cancel because the tensor products are over k, and the rest of the terms are manifestly elements of $D_n(B_*)$.

Similarly, for $r_n = \alpha$, the last two terms in the formula cancel, and the remaining terms are manifestly elements of $D_n(B_*)$. Thus, the correct interpretation of the formula for the boundary map is that if one uses a different representative for $\bar{r}_1$ or $\bar{r}_n$, the entire formula changes by an element of $D_n(B_*)$, which is considered trivial.

4.3 Right Bar Resolution of a Module.

Again, let $\gamma : k \to R$ be a ring homomorphism. Make the following definitions.

- $\mathcal{A}$ is the category of right k-modules.

- $\mathcal{B}$ is the category of right R-modules.
- $\mathcal{L} : \mathcal{A} \to \mathcal{B}$ is the base-change functor $A \mapsto A \otimes_k R$.
- $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ is the forgetful functor.

Then $(\mathcal{L}, \mathcal{R})$ is an adjoint pair of functors via the following maps:

- $\phi : \mathrm{Hom}_R(A \otimes R, B) \to \mathrm{Hom}_k(A, B)$ is the map sending each R-module map $f : R \otimes A \to B$ to the k-module map $\phi(f)$ sending $a \in A$ to $f(a \otimes 1)$.
- $\psi : \mathrm{Hom}_k(A, B) \to \mathrm{Hom}_R(R \otimes A, B)$ is the map sending each k-module map $g : A \to B$ to the R-module map $\psi(g) : A \otimes R \to B$ sending $a \otimes r$ to $g(a) \cdot r$.

To see this, first check that ϕ and ψ are inverse set maps. Indeed,

$$\psi\phi(f)(a \otimes r) = \phi(f)(a) \cdot r = f(a \otimes 1) \cdot r = f(a \otimes r)$$

since f is an R-module map, and

$$\phi\psi(g)(a) = \psi(g)(a \otimes 1) = g(a) \cdot 1 = g(a).$$

Next, check the naturality condition, which is the commutativity of the following diagram, where $h : A \to A'$ is a k-module map and $\ell : B \to B'$ is an R-module map:

Naturality Conditions for the Adjoint Pair $(\mathcal{L}, \mathcal{R}) = (- \otimes_k R, \mathcal{F})$.

$$\begin{array}{ccccc}
\mathrm{Hom}_R(A' \otimes R, B) & \xrightarrow{(h \otimes \mathrm{Id}_R)^*} & \mathrm{Hom}_R(A \otimes R, B) & \xrightarrow{\ell_*} & \mathrm{Hom}_R(A \otimes R, B') \\
\downarrow \phi & & \downarrow \phi & & \downarrow \phi \\
\mathrm{Hom}_k(A', B) & \xrightarrow{h^*} & \mathrm{Hom}_k(A, B) & \xrightarrow{\ell_*} & \mathrm{Hom}_k(A, B')
\end{array}$$

For the left square, suppose $f' \in \mathrm{Hom}_R(A' \otimes R, B)$. Then for any $a \in A$,

$$\phi \circ (h \otimes \mathrm{Id}_R)^*(f')(a) = (h \otimes \mathrm{Id}_R)^*(f')(a \otimes 1) = f'((h \otimes \mathrm{Id}_R)(a \otimes 1)) = f'(h(a) \otimes 1) = \phi(f')(h(a))$$

$$= h^* \circ \phi(f)(a)$$

For the right square, suppose $g \in \mathrm{Hom}_R(R \otimes A, B)$. Then for any $a \in A$,

$$\phi \circ \ell^*(g)(a) = \ell_*(g)(1 \otimes a) = \ell_* \circ \phi(g)(1 \otimes a).$$

$\mathrm{Hom}_R(A \otimes R, B)$ is a left k-module under the action $(\alpha \cdot f)(a \otimes r) = f(a \cdot \alpha \otimes r)$, and $\mathrm{Hom}_k(A, B)$ is a left k-module under the action $(\alpha \cdot g)(a) = g(a \cdot \alpha)$.

The maps ϕ and ψ are isomorphisms of left k-modules:

$$\phi(\alpha \cdot f)(a) = (\alpha \cdot f)(a \otimes 1) = f(a \cdot \alpha \otimes 1) = \phi(f)(a \cdot \alpha) = (\alpha \cdot \phi(f))(a),$$

and

$$\psi(\alpha \cdot \ g)(a \otimes r) = (\alpha \cdot g)(a) \cdot r = g(a \cdot \alpha) \cdot r = \psi(g)(a \cdot \alpha \otimes r) = (\alpha \cdot \psi(g))(a \otimes r).$$

Now define the cotriple $(\perp, \epsilon, \delta)$ of the map $\gamma : k \to R$ to be the cotriple of the adjoint pair $(\mathcal{L}, \mathcal{R})$ on the category $\mathcal{B}$ of R-modules. The functor $\perp = \mathcal{L}\mathcal{R}$ sends an R-module B to the R-module $B \otimes_k R$.

With this in mind,

- The unit η is the family of k-module maps η_A that send each k-module A into the underlying k-module of the R-module $A \otimes R$ via $a \mapsto a \otimes 1$. To see this, recall that for any element $f \in \mathrm{Hom}_R(A \otimes R, B)$, η_A is by definition the map $R \to A \otimes A$ such that $\phi(f) : A \to B$ is equal to $f \circ \eta_A$.
- The counit ϵ is the family of R-module maps ϵ_B that send the R-module $B \otimes R$ onto B via $b \otimes r \to b \cdot r$. To see this, recall that for any element $g \in \mathrm{Hom}_k(A, B)$, ϵ_B is by definition the map $B \otimes R \to B$ such that $\psi(g) : A \otimes R \to B$ is equal to $\epsilon_B \circ (g \otimes \mathrm{Id}_R)$.
- The natural transformation δ is the family of R-module maps sending $\perp B$ to $\perp^2 B$ via $b \otimes r \mapsto b \otimes 1 \otimes r$. This is obvious from the definition $\delta_B = \mathcal{L}\eta_{\mathcal{R}B}$.
- The natural transformation μ is the family of k-module maps sending $\top^2 A$ to $\top A$ via $a \otimes r \otimes r' \mapsto a \otimes rr'$. This is obvious from the definition $\mu_A = \mathcal{R}\epsilon_{\mathcal{L}A}$. Recall that $\top$ is the triple functor $\mathcal{R}\mathcal{L}$. The functor $\top$ and the "multiplication" μ are not crucial to this section.

For any R-module B, define B_* to be the corresponding simplicial object of the cotriple. The nth term B_n of B_* is $\perp^{n+1} B = B \otimes R^{\otimes (n+1)}$. The associated chain complex $C_*(B_*)$ is an aspherical resolution of B called the **right bar resolution**. The term "right" refers to the fact that B and B' are right R-modules.

It is useful to derive a more explicit description of the boundary maps in the right bar resolution, beginning with the face maps $d_n^i : B_n \to B_{n-1}$. By definition, these are

$$d_n^i = \perp^i \epsilon \perp^{n-i} : B_n = \perp^{n+1} B \longrightarrow \perp^n B = B_{n-1}.$$

First consider d_n^0, which is given by $\epsilon \perp^n$. This reduces to the map $\epsilon_{B_{n-1}} : B_n \to B_{n-1}$ given by the natural transformation $\epsilon : \perp \to \mathrm{Id}$. This map sends

$$b\otimes r_0\otimes r_1\otimes ...\otimes r_n\otimes r_n = (b\otimes r_0\otimes r_1\otimes ...\otimes r_{n-1})\otimes r_n \quad \text{to} \quad b\otimes r_0\otimes r_1\otimes ...\otimes r_{n-1}\cdot r_n = b\otimes r_0\otimes r_1\otimes ...\otimes r_{n-1}r_n.$$

Next, consider d_n^i for $0 < i \le n$, which is given by $\perp^i \epsilon \perp^{n-i}$, where the right-most exponent is at least one. First apply $\epsilon_{B_{n-i}}$ to the identity $B_{n-i} \to B_{n-i}$ to obtain the map sending $(b\otimes r_0\otimes ...\otimes r_{n-i})\otimes r_{n-i+1}$ to $b\otimes r_0\otimes ...\otimes r_{n-i}r_{n-i+1}$. Next, apply $\perp^i$ to obtain the map sending

$$b\otimes r_0\otimes ...\otimes r_n \qquad \text{to} \qquad b\otimes r_0\otimes ...\otimes r_{n-i}r_{n-i+1}\otimes ...\otimes r_n.$$

Finally, consider d_n^n, which is given by $\perp^n \epsilon$. First apply ϵ_B to the identity $B \to B$ to obtain the map sending $b\otimes r_0$ to $b\cdot r_0$. Next, apply $\perp^n$ to obtain the map sending

$$b\otimes r_0\otimes ...\otimes r_n \qquad \text{to} \qquad b\cdot r_0\otimes ...\otimes r_n.$$

Combining these formulas together with alternating signs, the boundary map $d_n : B_n \to B_{n-1}$ in the right bar resolution is given by

Boundary maps for the Right Bar Resolution of a right R-module B.

$$d_n(b\otimes r_0\otimes ...\otimes r_n) = \sum_{i=0}^{n}(-1)^i(b\otimes r_0\otimes ...\otimes r_{n-i}r_{n-i+1}\otimes ...\otimes r_n) + (-1)^n(b\cdot r_0\otimes ...\otimes r_n).$$

4.4 Normalized Right Bar Resolution of a Module.

Now consider the degeneracy maps $\sigma_n^i : B_n \to B_{n+1}$. By definition, these are

$$\sigma_n^i = \perp^i \delta \perp^{n-i} : B_n = \perp^{n+1} B \longrightarrow \perp^{n+2} B = B_{n+1}.$$

First consider σ_n^0, which is given by $\delta \perp^n$. This reduces to the map $\delta_{B_{n-1}} : B_n \to B_{n+1}$ given by the natural transformation $\delta : \perp \to \perp^2$. This map sends

$$b\otimes r_0\otimes ...\otimes r_n \qquad \text{to} \qquad b\otimes r_0\otimes ...\otimes r_{n-1}\otimes 1\otimes r_n$$

Similarly, σ_n^i sends

$$b\otimes r_0\otimes ...\otimes r_n \qquad \text{to} \qquad b\otimes r_0\otimes ...\otimes r_{n-i-1}\otimes 1\otimes r_{n-i}\otimes ...\otimes r_n$$

for $0 < i < n-1$. In particular, the last degeneracy map σ_n^n inserts a 1 between b and r_0.

The **degenerate right bar resolution** $D_*(B_*)$ is the complex generated by the images of the degeneracy maps. Its nth term $D_n(B_*)$ is generated by elements of the form $b \otimes r_0 \otimes r_1 ... \otimes r_n$, where at least one of the factors $r_0, ..., r_{n-1}$ in the tensor product is equal to 1. Note that since elements of k move across the tensor products, tensor product for which one of the last n factors is in k is an element of the degenerate complex.

The **normalized right bar resolution** $N_*(B_*)$ is the quotient $C_*(B_*)/D_*(B_*)$. Its nth term $N_n(B_*)$ is $B \otimes \bar{R}^{\otimes n} \otimes R$, generated by elements of the form $b \otimes \bar{r}_0 \otimes ... \otimes \bar{r}_{n-1} \otimes r_n$, where $\bar{r}_i$ is the class of $r_i \in R$ in the cokernel $\bar{R}$ of the map $k \to R$. The boundary maps in the normalized right bar resolution are induced by the boundary maps in the unnormalized right bar resolution:

Boundary maps for the Normalized Right Bar Resolution of a right R-module B.

$$d_n(b \otimes \bar{r}_0 \otimes ... \otimes \bar{r}_{n-1} \otimes r_n) = (b \cdot r_0 \otimes \bar{r}_1 \otimes ... \otimes \bar{r}_{n-1} \otimes r_n)$$
$$+ \sum_{i=1}^{n-1} (-1)^i (b \otimes \bar{r_0} \otimes ... \otimes \bar{r}_{n-i}\bar{r}_{n-i+1} \otimes ... \otimes \bar{r}_{n-1} \otimes r_n) + (-1)^n (b \otimes \bar{r}_0 \otimes ... \otimes \bar{r}_{n-2} \otimes r_{n-1} r_n).$$

4.5 Bar Resolution of a k-Algebra.

Again, let $\gamma : k \to R$ be a ring homomorphism, and consider the left bar resolution of the k-algebra R itself, viewed as a left R-module with the natural action of left multiplication. In fact, the left bar resolution is exactly the same as the right bar resolution in this case, so I use the notation $C_*^{bar}(R_*)$ to denote this resolution, and call it simply the **bar resolution of R**. The nth term is $C_n^{bar}(R_*) = R^{\otimes(n+1)} \otimes R = R^{\otimes(n+2)}$, and the boundary maps are given by the formula:

Boundary maps for the Bar Resolution of a k-Algebra R.

$$d_n(r_0 \otimes ... \otimes r_{n+1}) = \sum_{i=0}^{n} (-1)^i (r_0 \otimes ... \otimes r_i r_{i+1} \otimes ... \otimes r_{n+1})$$

4.6 Normalized Bar Resolution of a k-Algebra.

Similarly the normalized left bar resolution of R is the same as the normalized right bar resolution, so I use the notation $N_*^{bar}(R_*)$ to denote this resolution, and call it simply the **normalized bar**

resolution of R. The nth term is $N_n^{bar}(R_*) = R \otimes \bar{R}^{\otimes n} \otimes R$, and the boundary maps are given by the formula:

Boundary maps for the Normalized Bar Resolution of a k-Algebra R.

$$d_n(r_0 \otimes \bar{r}_1 \otimes \ldots \otimes \bar{r}_n \otimes r_{n+1}) = (r_0 r_1 \otimes \bar{r}_2 \otimes \ldots \otimes \bar{r}_n \otimes r_{n+1})$$
$$+ \sum_{i=0}^{n} (-1)^i (r_0 \otimes \bar{r}_1 \otimes \ldots \otimes \bar{r}_i \bar{r}_{i+1} \otimes \ldots \otimes \bar{r}_{n-1} \otimes r_{n+1}) + (r_0 \otimes \bar{r}_1 \otimes \ldots \otimes \bar{r}_{n-1} \otimes r_n r_{n+1})$$

5 Relative Ext Functors as Cotriple Cohomology Functors.

Let k be an associative ring and $\gamma : k \to R$ a ring homomorphism. Let B, B' be left R-modules. Let B_* be the simplicial object of the cotriple $\perp B = R \otimes_k B$, and let $C_*^{bar}(B_*)$ be the corresponding left bar resolution. Let $\mathcal{H} = \mathrm{Hom}_R(-, B')$ be the contravariant functor sending the left R-module B to the right R-module $\mathrm{Hom}_R(B, B')$, where the right R-module action on $\mathcal{H}(B) = \mathrm{Hom}_R(B, B')$ is given by the formula

$$(f \cdot \alpha)(b) = f(\alpha \cdot b).$$

The nth **relative Ext group** $\mathrm{Ext}^n_{R/k}(B, B')$ of the pair (B, B') with respect to the ring homomorphism $\gamma : k \to R$ is defined to be nth cotriple cohomology group of B with coefficients in the functor $\mathcal{H}$:

$$\mathrm{Ext}^n_{R/k}(B, B') := H^n(B, \mathcal{H}) = H^n(N^*(\mathcal{H}(B_*))) = H^n(N^*(\mathrm{Hom}_R(B_*, B'))).$$

Since $\mathcal{H}$ is a functor into an abelian category, this is the same as the cohomology of the unnormalized cochain complex $C^*(\mathcal{H}(B_*)) := C^*(\mathrm{Hom}_R(B_*, B'))$. This complex comes from applying the functor $\mathcal{H}$ to the left bar resolution $C_*^{bar}(B_*)$ of B, which is aspherical. Its nth term is $C^n(\mathcal{H}(B_*)) = \mathrm{Hom}_R(B_n, B')$, and its nth coface and codegeneracy maps are given by

$$d_i^n(f)(r_0 \otimes \ldots \otimes r_{n+1} \otimes b) = f \circ d_{n+1}^i(r_0 \otimes \ldots \otimes r_{n+1} \otimes b)$$

and

$$\sigma_i^n(f)(r_0 \otimes \ldots \otimes r_n \otimes b) = f \circ \sigma_{n-1}^i(r_0 \otimes \ldots \otimes r_n \otimes b).$$

Since $\mathcal{H} = \mathrm{Hom}_R(-, B')$ is left-exact, it follows that

$$\mathcal{H}(C_*^{bar}(B_*)) = C^*(\mathcal{H}(B_*)), \quad \mathcal{H}(N_*^{bar}(B_*)) = N^*(\mathcal{H}(B_*)), \quad \text{and} \quad \mathcal{H}(D_*^{bar}(B_*)) = D^*(\mathcal{H}(B_*)).$$

In particular, information about the cochain complexes $C^*(\mathcal{H}(B_*))$ and $N^*(\mathcal{H}(B_*))$, such as formulas for the corresponding coface, coboundary, and codegeneracy maps, may be analyzed by means of the left bar resolution $C_*^{bar}(B_*)$.

5.1 Coboundary Maps for $\mathbf{C}^*(\mathcal{H}(\mathbf{B}_*))$.

Begin with the unnormalized cochain complex $C^*(\mathcal{H}(B_*))$. The above expression for the coface maps d_i^n in terms of the face maps for the left bar resolution immediately leads to the following formulas:

$$d_i^n(f)(r_0 \otimes \ldots \otimes r_{n+1} \otimes b) = \begin{cases} r_0 \cdot f(r_1 \otimes \ldots \otimes r_{n+1} \otimes b) & i = 0; \\ f(r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_{n+1} \otimes b) & 1 \leq i \leq n; \\ f(r_0 \otimes \ldots \otimes r_{n+1} \cdot b) & i = n+1; \end{cases}$$

where the r_0 is pulled out of the expression $f(r_0 r_1 \otimes \ldots \otimes r_{n+1} \otimes b)$ for $d_0^n(f) = f \circ d_{n+1}^0$ because f is R-linear. This leads to the coboundary maps:

Coboundary maps for $C^*(\mathcal{H}(B_*))$; R-linear form.

$$d^n(f)(r_0 \otimes \ldots \otimes r_{n+1} \otimes b) = r_0 \cdot f(r_1 \otimes \ldots \otimes r_{n+1} \otimes b) + \sum_{i=1}^{n} (-1)^i f(r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_{n+1} \otimes b)$$
$$+(-1)^{n+1} f(r_0 \otimes \ldots \otimes r_n \otimes r_{n+1} \cdot b)$$

An alternative procedure allows for a slightly more compact description of the same maps. Note that since

$$\mathrm{Hom}_R(R \otimes A', B') \cong \mathrm{Hom}_k(A', B')$$

as right k-modules for any left k-module A' and left R-module B', it follows that

$$\mathrm{Hom}_R(B_n, B') = \mathrm{Hom}_R(R^{\otimes(n+1)} \otimes B, B') \cong \mathrm{Hom}_k(R^{\otimes n} \otimes B, B')$$

as right k-modules. Thus, the elements of $C^*(\mathcal{H}(B_*))$ may be viewed as k-multilinear maps, which are completely determined by their values on $(n+1)$-tuples of elements of the form $(r_1, \ldots, r_n, b)$.

In the previous description involving $\mathrm{Hom}_R(R^{\otimes(n+1)} \otimes B, B')$, the maps are R-linear, but *not* R-multilinear, because only elements of k move across the tensors.

Now return to the coface maps d_i^n, viewing them this time as k-linear maps on $\mathrm{Hom}_k(R^{\otimes n} \otimes B, B')$ rather than $\mathrm{Hom}_R(R^{\otimes(n+1)} \otimes B, B')$. This is accomplished by applying the isomorphisms ϕ and ψ that realize the adjunction between $\mathcal{H}$ and the forgetful functor $\mathcal{F}$. In other words, I will work now with $\phi \circ d_i^n \circ \psi$ instead of d_i^n, although I will refer to the composite map as d_i^n when there is no danger of confusion.

Beginning with a k-multlinear map $g \in \mathrm{Hom}_k(R^{\otimes n} \otimes B, B')$, consider the corresponding R-linear map $\psi(g) \in \mathrm{Hom}_R(R^{\otimes(n+1)} \otimes B, B')$ given by

$$\psi(g)(r_0 \otimes ... \otimes r_n \otimes b) = r_0 \cdot g(r_1, r_2, ..., r_n, b).$$

Next, calculate

$$d_0^n \circ \psi(g)(r_{-1} \otimes r_0 \otimes ... \otimes r_n \otimes b) = \psi(g) \circ d_{n+1}^0(r_{-1} \otimes r_0 \otimes ... \otimes r_n \otimes b) = \psi(g)(r_{-1}r_0 \otimes r_1 \otimes r_2 \otimes ... \otimes r_n \otimes b)$$

$$= r_{-1}r_0 \cdot g(r_1, r_2..., r_n, b)$$

Here I number from r_{-1} to r_n instead of from r_0 to r_{n+1} because the first argument disappears in the following calculations.

Now apply ϕ:

$$\phi \circ d_0^n \circ \psi(g)(r_0, r_1, r_2, ..., r_n, b) = d_0^n \circ \psi(g)(1 \otimes r_0 \otimes r_1 \otimes r_2 \otimes ... \otimes r_n \otimes b) = r_0 \cdot g(r_1, r_2..., r_n, b).$$

$$= r_0 \cdot g(r_1, r_2..., r_n, b),$$

by the previous calculation.

Next, consider d_i^n for $1 \leq i \leq n$. Compute $\phi \circ d_i^n \circ \psi(g)$:

$$\phi \circ d_i^n \circ \psi(g)(r_0, ..., r_n, b) = d_i^n \circ \psi(g)(1 \otimes r_0 \otimes ... \otimes r_n \otimes b) = \psi(g) \circ d_{n+1}^i(1 \otimes r_0 \otimes ... \otimes r_n \otimes b)$$

$$= \psi(g)(1 \otimes r_0 \otimes ... \otimes r_{i-1}r_i \otimes ... \otimes r_n \otimes b) = 1 \cdot g(r_0, ..., r_{i-1}r_i, ..., r_n, b)$$

$$= g(r_0, ..., r_{i-1}r_i, ..., r_n, b)$$

Finally, consider d^n_{n+1}. Compute $\phi \circ d^n_{n+1} \circ \psi(g)$:

$$\phi \circ d^n_{n+1} \circ \psi(g)(r_0, ..., r_n, b) = d^n_{n+1} \circ \psi(g)(1 \otimes r_0 \otimes, ... \otimes r_n \otimes b) = \psi(g) \circ d^{n+1}_{n+1}(1 \otimes r_0 \otimes ... \otimes r_n \otimes b)$$

$$= \psi(g)(1 \otimes r_0 \otimes ... \otimes r_{n-1} \otimes r_n \cdot b) = 1 \cdot g(r_0, ..., r_{n-1}, r_n \cdot b)$$

$$= g(r_0, ..., r_{n-1}, r_n \cdot b)$$

Combining these formulas together with alternating signs, and writing $d^n_i(g)$ for $\phi \circ d^n_i \circ \psi(g)$, the coboundary map $d^n : C^n(\mathcal{H}(B_*)) \to C^{n+1}(\mathcal{H}(B_*))$ sends a k-linear function g to the function $d^n(f)$ given by the formula:

Coboundary maps for $C^*(\mathcal{H}(B_*))$; k-linear form.

$$d^n(g)(r_0, ..., r_n, b) = r_0 \cdot g(r_1, ..., r_n, b) + \sum_{i=1}^{n} (-1)^i g(r_0, ..., r_{i-1}r_i, ..., r_n, b) + (-1)^{n+1} g(r_0, ..., r_n \cdot b).$$

5.2 The Normalized Complex $\mathbf{N}^*(\mathcal{H}(\mathbf{B}_*))$.

Suppose f is an element of $N^n(\mathcal{H}(B)) = \mathcal{H}(N^{bar}_n(B_*))$. I have shown that f may be regarded as an element of $\mathrm{Hom}_k(R^{\otimes n} \otimes B, B')$. Consider the value of f on an element of the form $(r_1, ..., r_i, 1, r_{i+2}, ..., r_n, b)$ for $1 \leq i \leq n$. For such an element,

$$f(r_1, ..., r_i, 1, r_{i+2}, ..., r_n, b) = f \circ \sigma^i_{n-1}(r_1, ..., r_i, r_{i+2}, ..., r_n, b) = 0,$$

since elements of $N^n(\mathcal{H}(B))$ vanish on elements the degenerate chain complex $D^{bar}_*(B_*)$.

Thus, the nth term degenerate complex may be regarded as the $N^n(\mathcal{H}(B))$ may be regarded as the k-module of k-multilinear maps $R^{\otimes n} \otimes B \to B'$ that vanish whenever one of the first n entries is an element of k.

6 Hochschild Cohomology in the Unital Case.

6.1 Definition in Terms of Relative Ext Groups.

Let $\gamma : k \to R$ be a homomorphism of unital rings. Let M be an R-bimodule. Let $R^e := R \otimes_k R^{op}$ be the **enveloping algebra** of R over k, where R^{op} is the **opposite algebra** of R, whose multiplication is given by defining the product ab in R^{op} to be equal to the product ba in R. The left action of R^e on M is given by $(r \otimes r') \cdot m = r \cdot m \cdot r'$, where the actions on the right side of the equation are the left and right R-module actions, respectively. Any R-bimodule is a left R^e-module with these actions, and the converse is also true by setting $r \cdot m = (r \otimes 1) \cdot m$ and $m \cdot r = (1 \otimes r) \cdot m$. The nth **Hochschild cohomology** group $HH^n(R, B)$ of R over k with coefficients in M is defined to be the nth relative Ext group:

$$HH^n(R, M) := \mathrm{Ext}^n_{R^e/k}(R, M).$$

Note that the Hochschild cohomology depends on the base ring k, even though k is suppressed in the notation $HH^n(R, M)$.

The definition of the relative Ext groups $\mathrm{Ext}^n_{R^e/k}(R, M)$ in terms of the left or right bar resolutions for the enveloping algebra R^e leads to a very cumbersome description of Hochschild cohomology, which is seldom if ever used in practice. Indeed, the unnormalized complex $\mathrm{Hom}_{R^e}(R_*, M)$, where R_* is the simplicial object of the cotriple $\perp B = R^e \otimes_k B$ for an arbitrary left R^e-module B, has coface maps beginning with

$$d_0^n(f)((r_0 \otimes r_0') \otimes ... \otimes (r_{n+1} \otimes r_{n+1}') \otimes r) = r_0 \cdot f((r_1 \otimes r_1') \otimes ... \otimes (r_{n+1} \otimes r_{n+1}') \otimes r) \cdot r_0',$$

where each tensor product $r_i \otimes r_i'$ is an element of R^e, and where the left and right R actions $r_0 \cdot f(-) \cdot r_0'$ on the right side of the formula express the left R^e-action $(r_0 \otimes r_0') \cdot f(-)$.

6.2 Description in terms of the Bar Complex.

A much more convenient description of Hochschild cohomology may be obtained by using the fact that relative Ext groups may be computed by means of k-split $\perp$-projective resolutions (see Weibel Section 8.7.1). In this case, the k-split $\perp$-projective resolution of interest is just the bar resolution $C_*^{bar}(R)$ of R as an R-module, where the elements and maps are regarded as being in the category of left R^e-modules, since R is an R-bimodule via left and right multiplication. To see that $C_*^{bar}(R)$ is indeed a k-split $\perp$-projective resolution of R, note that for any n, the map

$$\epsilon : \perp C_n^{bar}(R) = R^e \otimes R^{\otimes(n+2)} \to R^{\otimes(n+2)} = C_n^{bar}(R)$$

given by

$$(r \otimes r') \otimes (r_0 \otimes \ldots \otimes r_{n+1}) \mapsto r \cdot (r_0 \otimes \ldots \otimes r_{n+1}) \cdot r' = rr_0 \otimes \ldots \otimes r_{n+1}r'$$

has the obvious section

$$r_0 \otimes \ldots \otimes r_{n+1} \mapsto (1 \otimes 1) \otimes (r_0 \otimes \ldots \otimes r_{n+1}),$$

so that each $C_n^{bar}(R)$ is indeed a $\perp$-projective object. Note that the existence of this section depends on the existence of a multiplicative unit in R.

Also, each boundary map

$$d_n : (r_0 \otimes \ldots \otimes r_{n+1}) \mapsto \sum_{i=0}^{n} (-1)^i (r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_{n+1})$$

is split by the map $\sigma = \sigma_{n-1}^{-1}$, called the **extra degeneracy,** given by

$$\sigma : (r_0 \otimes \ldots \otimes r_n) \mapsto (1 \otimes r_0 \otimes \ldots \otimes r_{n+1})$$

"Splitting" in this context does *not* mean that $d_n \circ \sigma = \mathrm{Id}$, but only that $d_n \circ \sigma \circ d_n = d_n$. Note that the existence of this splitting again depends on the existence of a multiplicative unit.

From this it follows that

$$HH^n(R, M) := \mathrm{Ext}^n_{R^e/k}(R, M) = H^n(\mathrm{Hom}_{R^e}(C_*^{bar}(R), M)),$$

which is the definition given by Loday (although in his notation, he writes H_n on the right hand side rather than H^n.)

This formula leads to the coface maps $d_i^n(f) = f \circ d_{n+1}^i$, which reduce to

$$d_i^n(f)(r_0 \otimes \ldots \otimes r_n \otimes r_{n+1}) = \begin{cases} f(r_0 r_1 \otimes \ldots \otimes r_n \otimes r_{n+1}) & i = 0; \\ f(r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_n \otimes r_{n+1}) & 1 \le i \le n; \\ f(r_0 \otimes \ldots \otimes r_n \cdot r_{n+1}) & i = n+1; \end{cases}$$

Since each map $f \in \mathrm{Hom}_{R^e}(C_n^{bar}, M)$ may be viewed as a map of R-bimodules, the elements r_0 and r_{n+1} may be pulled out of the formulas for $d_0^n(f)$ and $d_n^n(f)$ via the left and right actions of R. This yields the following expression for the coboundary map:

Hochschild Coboundary maps; R-linear form.

$$d^n(f)(r_0 \otimes \ldots \otimes r_{n+1}) = r_0 \cdot f(r_1 \otimes \ldots \otimes r_{n+1}) + \sum_{i=1}^{n} (-1)^i f(r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_{n+1})$$
$$+(-1)^{n+1} f(r_0 \otimes \ldots \otimes r_n) \cdot r_{n+1}$$

6.3 Description in terms of k-multilinear maps.

In previous sections, I used the fact that

$$\mathrm{Hom}_R(R \otimes A', B') \cong \mathrm{Hom}_k(A', B')$$

to give a simpler version of the cochain complex $C^*(\mathrm{Hom}_R(B_*, B'))$ whose cohomology groups are the relative Ext groups $\mathrm{Ext}^n_{R/k}(B, B')$. A similar procedure yields a simpler version of the cochain complex $\mathrm{Hom}_{R^e}(C^{bar}_*(R), M) = C^*(\mathrm{Hom}_{R^e}(R^{\otimes *}, M))$ whose cohomology groups are the Hochschild groups $HH^n(R, M) \coloneqq \mathrm{Ext}^n_{R^e/k}(R, M)$.

First note that $R^e \otimes R^{\otimes n} \cong R \otimes (R^{\otimes n}) \otimes R = R^{\otimes (n+2)}$ as left R^e-modules (equivalently, as R-bimodules) via the map

$$(r \otimes r') \otimes (r_1 \otimes \ldots \otimes r_n) \mapsto r \otimes r_0 \otimes \ldots \otimes r_n \otimes r'.$$

Next, for any R-bimodule M,

$$\mathrm{Hom}_{R^e}(R^{\otimes (n+2)}, M) \cong \mathrm{Hom}_{R^e}(R^e \otimes R^{\otimes n}, M) \cong \mathrm{Hom}_k(R^{\otimes n}, M)$$

as k-modules, by the adjointness of the functor $N \mapsto R^e \otimes_k N$ and the forgetful functor from R^e-modules to k-modules. The isomorphism is the map $\phi : \mathrm{Hom}_{R^e}(R^{\otimes (n+2)}, M) \to \mathrm{Hom}_k(R^{\otimes n}, M)$ given by

$$\phi(f)(r_1, \ldots, r_n) = f(1 \otimes r_1 \otimes \ldots \otimes r_n \otimes 1),$$

and the inverse is the map $\psi : \mathrm{Hom}_{R^e}(R^{\otimes n}, M) \to \mathrm{Hom}_k(R^{\otimes (n+2)}, M)$ given by

$$\psi(g)(r \otimes r_1 \otimes \ldots \otimes r_n \otimes r') = r \cdot \psi(g)(1 \otimes r_1 \otimes \ldots \otimes r_n \otimes 1) \cdot r' = r \cdot g(r_1, \ldots, r_n) \cdot r'.$$

These isomorphisms permit a simpler description of the Hochschild coface maps d^n_i. First consider $\phi \circ d^n_0 \circ \psi$:

$$\phi \circ d_0^n \circ \psi(g)(r_1, ..., r_{n+1}) = d_0^n \circ \psi(g)(1 \otimes r_1 \otimes ... \otimes r_{n+1} \otimes 1) = \psi(g) \circ d_{n+1}^0(1 \otimes r_1 \otimes ... \otimes r_{n+1} \otimes 1)$$

$$= \psi(g)(r_1 \otimes ... \otimes r_{n+1} \otimes 1) = r_1 \cdot g(r_2, r_3, ..., r_{n+1}) \cdot 1$$

$$= r_1 \cdot g(r_2, r_3, ..., r_{n+1}).$$

Next, consider $\phi \circ d_i^n \circ \psi$ for $1 \leq i \leq n$:

$$\phi \circ d_i^n \circ \psi(g)(r_1, ..., r_{n+1}) = d_i^n \circ \psi(g)(1 \otimes r_1 \otimes ... \otimes r_{n+1} \otimes 1) = \psi(g) \circ d_{n+1}^i(1 \otimes r_1 \otimes ... \otimes r_{n+1} \otimes 1)$$

$$= \psi(g)(1 \otimes r_1 \otimes ... \otimes r_i r_{i+1} \otimes ... \otimes r_{n+1} \otimes 1) = 1 \cdot g(r_1, ..., r_i r_{i+1}, ..., r_{n+1}) \cdot 1$$

$$= g(r_1, ..., r_i r_{i+1}, ..., r_{n+1}).$$

Finally, consider $\phi \circ d_{n+1}^n \circ \psi$:

$$\phi \circ d_{n+1}^n \circ \psi(g)(r_1, ..., r_{n+1}) = d_{n+1}^n \circ \psi(g)(1 \otimes r_1 \otimes ... \otimes r_{n+1} \otimes 1) = \psi(g) \circ d_{n+1}^{n+1}(1 \otimes r_1 \otimes ... \otimes r_{n+1} \otimes 1)$$

$$= \psi(g)(1 \otimes r_1 \otimes ... \otimes r_i r_{i+1} \otimes ... \otimes r_{n+1}) = 1 \cdot \psi(g)(r_1, ..., r_n) \cdot r_{n+1}$$

$$= \psi(g)(r_1, ..., r_n) \cdot r_{n+1}.$$

Combining these formulas together with alternating signs, the Hochschild boundary maps given by the formula:

Hochschild Coboundary maps; k-linear form.

$$d^n(g)(r_1, ..., r_{n+1}) = r_1 \cdot f(r_2, ..., r_{n+1}) + \sum_{i=1}^{n} (-1)^i f(r_1, ..., r_i r_{i+1}, ..., r_{n+1})$$
$$+(-1)^{n+1} f(r_1, .., r_n) \cdot r_{n+1}$$

6.4 Description in terms of Absolute Ext Groups for Projective Algebras.

The absolute Ext groups $\mathrm{Ext}^n_{R^e}(R, M)$ may be computed via an arbitrary projective R^e-module resolution $P_* \to R$ of R. Indeed, for such a resolution,

$$\mathrm{Ext}^n_{R^e}(R, M) = H^n(\mathrm{Hom}_{R^e}(P_*, M))$$

by elementary homological algebra.

I showed above that the bar complex $C^{bar}_*(R)$ is a k-split $\perp$-projective resolution of R as an R^e-module, where $\perp$ is the cotriple functor $\mathcal{L}\mathcal{R}$ given by composing the functor $\mathcal{L} : M \mapsto R \otimes_k M$ with the forgetful functor $\mathcal{R}$ from the category of R-modules to the category of k-modules. However, this does not require that the algebra R is projective as a k-module; it is true in general.

Suppose, however, R *is* projective as a k-module. Then $R^{\otimes n}$ is projective as an k-module; see Weibel Chapter 9 Lemma 9.1.4 page 303 for a proof. Thus, $C^{bar}_n(R) \cong R^e \otimes R^{\otimes n}$ is projective as a k-module, and hence as an R^e-module. This is true because the condition that $C^{bar}_n(R)$ is projective as a k-module is equivalent to the condition that the functor $\mathrm{Hom}_k(C^{bar}_n(R), -)$ is exact, which implies that the functor $\mathrm{Hom}_{R^e}(C^{bar}_n(R), -)$ is exact, since every R^e-module is a k-module. Therefore the bar complex $C^{bar}_*(R)$ is a resolution of R by projective R^e-modules, so the absolute Ext group $\mathrm{Ext}^n_{R^e}(R, M)$ is given by:

$$\mathrm{Ext}^n_{R^e}(R, M) = H^n(\mathrm{Hom}_{R^e}(C^{bar}_*(R), M))$$

This is the formula for the relative Ext group $\mathrm{Ext}^n_{R^e/k}(R, M)$, which by definition is the Hochschild cohomology $HH^n(R, M)$, in terms of the bar complex. Therefore,

$$HH^n(R, M) = \mathrm{Ext}^n_{R^e}(R, M)$$

if R is projective as a k-module.

7 Algebras in More General Settings.

An **algebra over a ring** k is a k-bimodule R equipped with **product map** $\mu : R \otimes_k R \to R$ which is left k-linear in the first factor and right k-linear in the second factor. That is,

$$\mu\Big((\alpha r + \alpha' r') \otimes (s\beta + s'\beta')\Big) = \alpha\mu(r \otimes s)\beta + \alpha\mu(r \otimes s')\beta' + \alpha'\mu(r' \otimes s)\beta + \alpha'\mu(r' \otimes s')\beta'.$$

for $\alpha, \alpha', \beta, \beta' \in k$ and $r, r', s, s' \in R$. Here the k-bimodule action is denoted by juxtaposition; e.g. αr is short for $\alpha \cdot r$. The same information could be expressed by taking μ to be a map from the cartesian product $R \times R$ to R, but this would require additional "middle-linearity conditions" that are satisfied automatically by the tensor product.

R is a ring under the product map μ because μ distributes over addition. Note that rings are often defined with an assumption of associativity; i.e., a semigroup structure under multiplication. I do *not* make this assumption in general. In this setting, neither k nor R need be unital, commutative, or associative.

If R is unital with unit 1_R, then for any $\alpha \in k$,

$$\alpha \cdot 1_R = \mu(1_R \otimes \alpha \cdot 1_R) = \mu(1_R \cdot \alpha \otimes 1_R) = 1_R \cdot \alpha.$$

Define a map $u : k \to R$ by setting $u(\alpha) = \alpha \cdot 1_R = 1_R \cdot \alpha$ for any $\alpha \in k$. Then

$$u(\alpha + \beta) = (\alpha + \beta) \cdot 1_R = \alpha \cdot 1_R + \beta \cdot 1_R = u(\alpha) + u(\beta)$$

and

$$u(\alpha\beta) = (\alpha\beta) \cdot 1_R = \alpha \cdot (\beta \cdot 1_R) = \mu(1_R \otimes \alpha \cdot (\beta \cdot 1_R)) = \mu(1_R \cdot \alpha \otimes \beta \cdot 1_R) = \mu(\alpha \cdot 1_R \otimes \beta \cdot 1_R)$$

$$= \mu(u(\alpha) \otimes u(\beta)).$$

Thus, if R is unital, then $u : k \to R$ is a homomorphism of (generally nonassociative) rings. Conversely, a ring homomorphism $\gamma : k \to R$ induces a k-bimodule structure on R, and multiplication on R automatically satisfies the k-linearity conditions for R to be a k-algebra. Thus, existence of a unit in a k-algebra R is a sufficient but generally not necessary condition for the existence of a ring homomorphism $k \to R$ inducing the underlying bimodule structure.

Many of the examples of cotriples do not generalize to the nonunital case, since the corresponding pairs of functors are not adjoint in this case. For example, let $k = \mathbb{Z}$, and let R be the nonunital ring $2\mathbb{Z}$. Then multiplication by 2 defines a ring homomorphism $k \to R$. Let $A = B = \mathbb{Z}_2$ with k-action given my multiplication modulo 2, and trivial R-action (since every multiple of 2 is trivial in $\mathbb{Z}_2$). With these definitions, the Hom-sets

$$\mathrm{Hom}_R(R \otimes_k A, B) = \mathrm{Hom}_{2\mathbb{Z}}(2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2, \mathbb{Z}_2) \quad \text{and} \quad \mathrm{Hom}_k(A, B) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2)$$

are *not* bijective. Indeed, $R \otimes_k A$ is trivial in this case, and the only additive map from the trivial module to $\mathbb{Z}_2$ is the zero map. Meanwhile, $\mathrm{Hom}_k(A, B)$ has two elements, the zero map and the identity.

8 Hochschild Cohomology in the Nonunital Case.

If R is an associative but nonunital k-algebra, the complex of R^e-linear maps $\mathrm{Hom}_{R^e}(R^{\otimes(*+2)}, M)$ with coboundary maps

$$d^n(f)(r_0 \otimes \ldots \otimes r_{n+1}) = r_0 \cdot f(r_1 \otimes \ldots \otimes r_{n+1}) + \sum_{i=1}^{n}(-1)^i f(r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_{n+1})$$

$$+(-1)^{n+1} f(r_0 \otimes \ldots \otimes r_n) \cdot r_{n+1}$$

and the complex of k-multilinear maps $\mathrm{Hom}_k(R^{\otimes *}, M)$ with coboundary maps

$$d^n(g)(r_1, \ldots, r_{n+1}) = r_1 \cdot f(r_2, \ldots, r_{n+1}) + \sum_{i=1}^{n}(-1)^i f(r_1, \ldots, r_i r_{i+1}, \ldots, r_{n+1})$$
$$+(-1)^{n+1} f(r_1, .., r_n) \cdot r_{n+1}$$

are still cochain complexes (by direct calculation), but are no longer isomorphic in general, and do not arise from a cotriple in any obvious way. I do not know whether their cohomology groups are isomorphic, nor do I know if there is some cotriple or cotriples to which they correspond.

In any case, there is a standard way of extending a functor $\mathcal{F}$ from the category of unital algebras to abelian groups to include nonunital algebras. This is how Hochschild cohomology is defined in the nonunital case. Given a non-unital k-algebra R, begin by defining a unital k-algebra R_+ associated to R by taking the underlying k-module structure to be $k \oplus R$ and defining multiplication by the formula

$$(\alpha, r)(\alpha', r') := (\alpha\alpha', \alpha r' + \alpha' r + r r')$$

The unit in R_+ is then $(1_k, 0)$. With this definition, there is a canonical algebra map $k \to R_+$ defined by sending $\alpha \in k$ to $(\alpha, 0)$ in R_+. For any functor $\mathcal{F}$ from unital algebras to abelian groups, apply $\mathcal{F}$ to this canonical algebra map, and define

$$\mathcal{F}(R) := \mathrm{coker}[\mathcal{F}(k) \to \mathcal{F}(R_+)]$$

In particular, the **Hochschild cohomology** $HH^n(R, M)$ is defined by

$$HH^n(R, M) := \mathrm{coker}[HH^n(k, M) \to HH^n(R_+, M)].$$