Infinitesimal Theory of Chow Groups

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Introduction

- The Chow groups Ch_X^p for a smooth algebraic variety X over a field k are central to algebraic geometry.
- Poorly understood for $p \geq 2$.
- Green-Griffiths 2005 [1]: **"Study their tangent groups!"**
- Dribus-Hoffman-Yang have improved this approach:
  - New infinitesimal theory of Ch_X^p.
  - Generalized tangent groups $T_Y\mathrm{Ch}_X^p$.
- Depends on:
  - Bass-Thomason algebraic K-theory.
  - Negative cyclic homology.
  - Coniveau spectral sequence.
  - Algebraic Chern character.

Notation and Conventions

- Distinguished technical terms are blue: Chow groups.
- Math symbols are red: Ch^p_X.
- New concepts and results are green: coniveau machine.
- Reference hyperlinks are grey: Bloch 1972 [8].
- Parentheses are often suppressed: Ch^p_X, not $\mathrm{Ch}^p(X)$.

Where We're Headed I

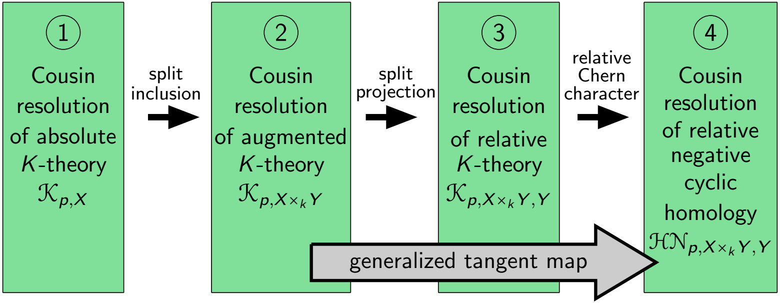
Overarching goal: define computable invariants involving algebraic cycles, and show how to compute them!

- **Coniveau machine:** special diagram of functors and natural transformations defined for this purpose.
- **Theorem (DHY, 2012):** Coniveau machine for Chow groups exists.
- **Corollary:** Can compute generalized tangent groups of Chow groups via negative cyclic homology:

$$T_Y \mathrm{Ch}_X^p \cong H_{\mathrm{Zar}}^p\big(X, \mathcal{HN}_{p, X\times_k Y, Y}\big).$$

Where We're Headed II

- Schematic diagram of coniveau machine:

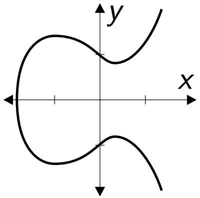


- Generalized tangent groups $T_Y\mathrm{Ch}_X^p$ computed via fourth column.
- Generalized tangent map takes "deformations" to their "tangent elements."

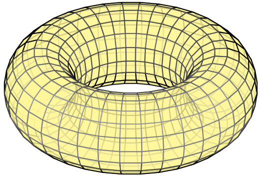
Algebraic Varieties

- Technically, an algebraic variety X is an integral, separated scheme of finite type over an algebraically closed field k.
- Informally, X defined locally by vanishing of polynomials.
- Example: X the complex algebraic curve $y^2 = x^3 + x^2 - x + 1$:

real points of X



X as a Riemann surface

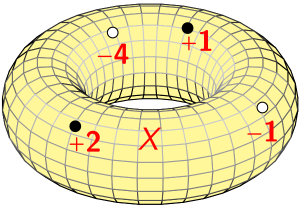


Algebraic Cycles

- Codimension-p algebraic cycle z on X:
  - Formal linear combination of codimension-p subvarieties of X.
  - Written as a finite formal sum:

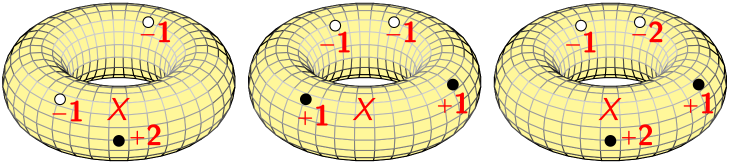
$$z = \sum_{x \in \mathrm{Zar}_X^p} n_x x.$$

  - Subvarieties are labeled by their generic points x.
  - Zar_X^p: set of codimension-p points in Zariski topology on X.
  - The n_x are multiplicities, usually integers.
- Example: codimension-one cycle on an algebraic curve X:



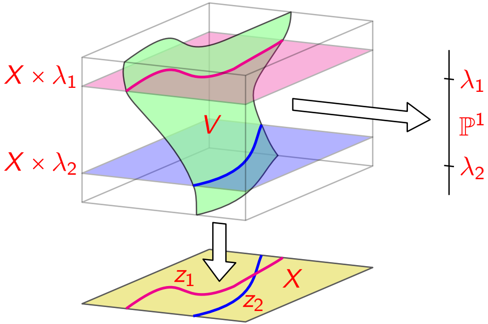
Cycle Groups

- Z_X^p: set of codimension-p cycles on X.
  - Z_X^p an abelian group called pth cycle group of X.
  - Group operation on Z_X^p induced by addition of multiplicities.
- Example: addition of codimension-one cycles on a curve:



Rational Equivalence

- Cycle groups are "huge," and have poor intersection theory.
- Strategy: impose an adequate equivalence relation.
    - Introduced by Samuel 1958 [2].
    - Rational equivalence: two cycles "belong to a $\mathbb{P}^1$-family."



- Group of rational equivalence classes of codimension-p cycles is the pth Chow group Ch^p_X of X.

"Easy Case:" Divisors and Ch_X^1

- Divisors: codimension-1 cycles.
- Examples: points on a curve, curves on a surface.
- "Easy" because:
  - $\mathrm{Ch}_X^1 \cong \mathrm{Pic}_X$ (Picard group, an algebraic group.)
  - Algebraic and homological equivalence coincide.
  - Gives Picard variety Pic_X^0, an abelian variety.
  - Example: $X \cong \mathrm{Pic}_X^0$ for genus-1 curve X, via Abel-Jacobi map.
- Lefschetz $(1,1)$-theorem:
  - $H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ consists of divisor classes.
  - Special case of Hodge Conjecture.

Ch_X^p for Higher Codimension

- For $p \geq 2$, Ch_X^p remains poorly understood.
- Famous results make situation look worse, not better:
  - Mumford 1969 [3]:

$$\dim \mathrm{Ch}_X^2 = \infty \qquad \text{(points on a surface)}$$

  - Griffiths 1969 [4]:

$$\text{alg. equ.} \neq \text{hom. equ.} \qquad \text{(curves on a threefold)}$$

  - Clemens 1983 [5]:

$$\dim \frac{\text{alg. equ.}}{\text{hom. equ.}} = \infty \qquad \text{(codimension 2)}$$

- Conjectures: Hodge conjecture, Tate conjecture, conjectured filtrations of Ch_X^p, etc.

Approach of Green and Griffiths

- Strategy: **“Linearize the problem!”**
  - Conceptually similar to Lie theory: linearize Lie group to obtain Lie algebra.
  - Uses algebraic K-theory.
- Previous/contemporary related work:
  - Van der Kallen [6]: tangent space of K_2.
  - Bloch [7], [8]: Ch_X^2 via K-theory; tangent space of Ch_X^2.
  - Quillen [9]: extensions via higher K-theory.
  - Stienstra [10], [11], [12]: Cartier-Dieudonné theory for Chow groups (very sophisticated)
  - Hesselholt [13]: K-theory of truncated polynomial algebras.
  - Many others!

Bloch's Formula

- Bloch-Quillen 1972 [7], [9]:

$$\mathrm{Ch}_X^p \cong H_{\mathrm{Zar}}^p(X, \mathcal{K}_{p,X}).$$

(Zariski sheaf cohomology.)

- Computed via following sheafified Cousin complex:

$$0 \longrightarrow \mathcal{K}_{p,X} \longrightarrow \coprod_{x \in \mathrm{Zar}_X^0} \underline{K_p(k_x)} \longrightarrow \coprod_{x \in \mathrm{Zar}_X^1} \underline{K_{p-1}(k_x)} \longrightarrow \cdots \cdots \longrightarrow \coprod_{x \in \mathrm{Zar}_X^p} \underline{K_0(k_x)} \longrightarrow 0$$

  - Called Bloch-Gersten-Quillen resolution (BGQR) of $\mathcal{K}_{p,X}$.
  - Milnor version: use Milnor K-theory $\mathcal{K}_{p,X}^{\mathrm{M}}$ instead.

  (Kerz 2006 [14]; previously known only up to torsion!)

Green and Griffiths' Tangent Group $T_{\mathrm{GG}}\mathrm{Ch}^p_X$

- Green and Griffiths 2005 [1] define tangent group at the identity of Ch^p_X via Milnor K-theory:

$$T_{\mathrm{GG}}\mathrm{Ch}^p_X := H^p_{\mathrm{Zar}}(X, T\mathcal{K}^{\mathrm{M}}_{p,X}) = H^p_{\mathrm{Zar}}(X, \Omega^{p-1}_{X/\mathbb{Q}}).$$

  - $T\mathcal{K}^{\mathrm{M}}_{p,X}$: tangent sheaf of $\mathcal{K}^{\mathrm{M}}_{p,X}$.
  - $\Omega^{p-1}_{X/\mathbb{Q}}$: sheaf of absolute Kähler differentials.
  - $T\mathcal{K}^{\mathrm{M}}_{p,X} \cong \Omega^{p-1}_{X/\mathbb{Q}}$: primitive relative algebraic Chern character.
  - $T_{\mathrm{GG}}\mathrm{Ch}^p_X$: my notation.
- Brings arithmetic considerations to the forefront, even for complex varieties.

Tangent Sequences and Sheaves

- Green-Griffiths' focus: X a smooth algebraic surface.
- Identify the tangent sequence to the BGQR of $\mathcal{K}_{2,X}$ as the following sheafified Cousin complex:

$$0 \longrightarrow \Omega^1_{X/\mathbb{Q}} \longrightarrow \coprod_{x \in \mathrm{Zar}^0_X} \underline{H^0_x(\Omega^1_{X/\mathbb{Q}})} \longrightarrow \coprod_{x \in \mathrm{Zar}^1_X} \underline{H^1_x(\Omega^1_{X/\mathbb{Q}})} \longrightarrow \coprod_{x \in \mathrm{Zar}^2_X} \underline{H^2_x(\Omega^1_{X/\mathbb{Q}})} \longrightarrow 0$$

  - Called Cousin flasque resolution (CFR) of $\Omega^1_{X/\mathbb{Q}}$.
  - $H^q_x(\Omega^1_{X/\mathbb{Q}})$ is local cohomology (Hartshorne 1966 [16]).

Computing Ch^2_X and $T_{\mathrm{GG}}\mathrm{Ch}^2_X$

- Consider the final maps d_1 and Td_1 in the BGQR of $\mathcal{K}_{X,2}$ and the CFR of $\Omega^1_{X/\mathbb{Q}}$:

$$\coprod_{x\in \mathrm{Zar}^1_X} \underline{K_1(k_x)} \xrightarrow{d_1} \coprod_{x\in \mathrm{Zar}^2_X} \underline{K_0(k_x)} \longrightarrow 0$$

$$\coprod_{x\in \mathrm{Zar}^1_X} \underline{H^1_x(\Omega^1_{X/\mathbb{Q}})} \xrightarrow{Td_1} \coprod_{x\in \mathrm{Zar}^2_X} \underline{H^2_x(\Omega^1_{X/\mathbb{Q}})} \longrightarrow 0$$

- By the definition of sheaf cohomology:

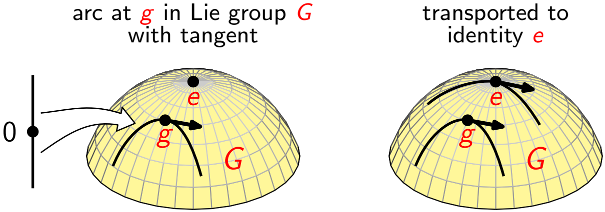
$$\mathrm{Ch}^2_X \cong H^2_{\mathrm{Zar}}(X, \mathcal{K}_{2,X}) = \frac{\Gamma \coprod_{x\in \mathrm{Zar}^2_X} \underline{K_0(k_x)}}{\mathrm{Im}\,\Gamma d_1}$$

- Similarly, using Green-Griffiths' definition:

$$T_{\mathrm{GG}}\mathrm{Ch}^2_X = H^2_{\mathrm{Zar}}(X, \Omega^1_{X/\mathbb{Q}}) = \frac{\Gamma \coprod_{x\in \mathrm{Zar}^2_X} \underline{H^2_x(\Omega^1_{X/\mathbb{Q}})}}{\mathrm{Im}\,\Gamma Td_1}$$

Geometric Interpretation

- Green-Griffiths provide geometric interpretation for TCh^2_X in terms of "arcs" of zero cycles and arcs of rational equivalences.
- "Arcs" are defined only informally.
- The idea is familiar from differential geometry and Lie theory:



Improving on GG

GG's work is presented as exploratory, not definitive.

- Conceptual framework is unclear.
- Rigor is lacking.
- Difficult to apply, or even assess.
- Better tools available: Bass-Thomason K-theory, cyclic homology, algebraic Chern character, etc.
- Recent results offer improvements (e.g. Kerz 2006 [14], Cortiñas et al. 2008 [15]).
- Vast generalizations are possible.

Introducing Our Approach

Our approach is based on four general ideas:

1. Cohomology theories with supports (CTS).
2. Coniveau filtration of topological spaces, and the resulting coniveau spectral sequences (CSS).
3. Nilpotent augmentation.
4. Exponential and logarithmic-type maps.

Cohomology Theories with Supports

A cohomology theory with supports (CSS) is a special family of functors.

- Source: distinguished category **S** of topological spaces (e.g., separated schemes over a field k).
- Target: abelian category **A**.
- Respects topological structure in a special way.
- Often arises from a spectrum-valued functor (substratum).
- Examples:
  - Bass-Thomason algebraic K-theory via substratum **K**.
  - Negative cyclic homology via substratum **HN**.
  - Many others (Colliot-Thélène, Hoobler, and Kahn 1997 [17]).
- Non-examples: Quillen K-theory, Milnor K-theory.

Coniveau Filtration and Coniveau Spectral Sequence

- Coniveau filtration organizes the points of a topological space by codimension.
  - "Coniveau" means "codimension" (French).
  - Example: Zariski topology on n-dimensional scheme:

$$\varnothing \subset \mathrm{Zar}_X^{\geq n} \subset \mathrm{Zar}_X^{\geq n-1} \subset \ldots \subset \mathrm{Zar}_X^{\geq 1} \subset \mathrm{Zar}_X^{\geq 0} = \mathrm{Zar}_X$$

  - Very general method.
- Coniveau spectral sequence (CSS) for a CTS on a category $\mathbf{S}$:
  - Induced by coniveau filtration via exact couple.
  - Rows are Cousin complexes.
  - Examples: BGQR of $\mathcal{K}_{2,X}$ and CFR of $\Omega^1_{X/\mathbb{Q}}$ are sheafified versions of Cousin complexes.

Nilpotent Augmentation

- Algebraic notion of "infinitesimal degrees of freedom."
- Simplest example: $R \mapsto R_\varepsilon := R[\varepsilon]/\varepsilon^2$.
  - Idea: ε is "so small that $\varepsilon^2 = 0$."
  - If R is a k-algebra, then $R_\varepsilon = R \otimes_k k_\varepsilon$.
  - k_ε: "algebra of dual numbers."
  - For k-schemes: $X \mapsto X \times_k k_\varepsilon$ (fiber product).
  - First-order infinitesimal theory; gives tangent space.
- More general: replace k_ε:
  - Use any k-algebra A such that $\mathrm{Ker}[A \to k]$ is nilpotent.
  - For k-schemes: $X \mapsto X \times_k Y$, where $Y = \mathrm{Spec}\ A$.
  - Higher-order theory; gives generalized tangent space.
- Does not affect topological structure!

Exponential and Logarithmic-Type Maps

- Ubiquitous in mathematics; e.g., Lie theory.
- Algebraic setting: algebraic Chern character.
    - Primitive version: dlog map from Milnor K-theory to differential forms.
    - Maps algebraic K-theory to negative cyclic homology.
    - May be viewed as natural transformation of functors:

$$\mathrm{ch} : \mathbf{K} \mapsto \mathbf{HN}.$$

    - Relative version induces isomorphisms for nilpotent augmentations:

$$\mathrm{ch}_p : \mathcal{K}_{p,X\times_k Y,Y} \to \mathcal{HN}_{p,X\times_k Y,Y}$$

    - Related to Goodwillie's isomorphism [18] (sheafified):

$$\rho_p : \mathcal{K}_{n,X\times_k Y,Y} \otimes \mathbb{Q} \to \mathcal{HC}_{p-1,X\times_k Y,Y} \otimes \mathbb{Q}$$

Our Approach to Chow Groups

- Use Bass-Thomason K-theory, not Milnor K-theory.
  - View $X \mapsto H^p_{\mathrm{Zar}}(X, \mathcal{K}_{p,X})$ as extension of Chow functor.
  - Treat smooth and nilpotent-augmented structure functorially on same footing (Bass-Thomason K-theory essential).
  - View BGQR of $\mathcal{K}_{p,X}$ as $-p$th sheafified Cousin complex from CSS for $\mathbf{K}$.
- Identify tangent sequence of BGQR as CFR of relative negative cyclic homology $\mathcal{HN}_{p,X\times_k Y,Y}$.
- Define tangent map via relative algebraic Chern character:

$$\mathrm{ch}_{p,X\times_k Y,Y} : \mathcal{K}_{p,X\times_k Y,Y} \to \mathcal{HN}_{p,X\times_k Y,Y}.$$

Generalizing Our Approach I

- Whenever a family of functors $H = \{H^n\}_{n\in\mathbb{Z}}$ sufficiently respects topological structure, Chow group analogues exist.
- H must:
  - Take values in an abelian category.
  - Produce long exact sequences for inclusions of closed subsets.
  - Satisfy an appropriate local condition (effacement).
- Such a family H is called an effaceable cohomology theory with supports (CTS).
  - **K** and **HN** are examples.
  - Colliot-Thélène, Hoobler, and Kahn 1997 [17] mention others.
  - The CSS of an effaceable CTS H yields CFR's of the associated sheaves $\mathcal{H}_X^p$.

Generalizing Our Approach II

- To admit nilpotent augmentation, a CTS H must be mediated by ring structure.
- Example: H is Bass-Thomason algebraic K-theory:
  - $\mathcal{K}_{p,X}$ is the sheaf associated to the presheaf $U \mapsto K_{p,O_U}$.
  - O_U is the ring of sections of $\mathcal{O}_X$ over U.
  - Nilpotent augmentation occurs at level of O_U.
  - E.g., for schemes over a field k, $O_U \mapsto O_U \otimes_k A$, where A is generated by nilpotents.
- Hence, ringed spaces are the natural setting for nilpotent structure, though the CSS is purely topological.
- Generalized tangent maps are natural transformations from augmented to relative CTS's.

Extending the Chow Functors

- Chow functor $X \mapsto \mathrm{Ch}_X^p$ is purely topological.
- Must extend it to "see nilpotent structure."
- Two possible choices:
  1. Bloch-Milnor: $X \mapsto H_{\mathrm{Zar}}^p(X, \mathcal{K}_{p,X}^{\mathrm{M}})$.
  2. Bloch: $X \mapsto H_{\mathrm{Zar}}^p(X, \mathcal{K}_{p,X})$.
- Same for X smooth.
- **Different in general!**

Defining Tangent Groups

- Goal: define tangent groups at the identity of Ch_X^p.
  - Analogous to Lie algebras.
  - Depends on choice of extension of Chow functor.
- Green-Griffiths [1] use Bloch-Milnor to define tangent group:

$$T_{\mathrm{GG}}\mathrm{Ch}_X^p := H_{\mathrm{Zar}}^p(X, T\mathcal{K}_{p,X}^{\mathrm{M}}).$$

- We (Dribus-Hoffman-Yang) use Bloch to define generalized tangent groups:

$$T_Y\mathrm{Ch}_X^p := H_{\mathrm{Zar}}^p(X, \mathcal{K}_{p,X\times_k Y,Y}).$$

Why Our Definition of $T_Y \mathrm{Ch}_X^p$?

- Algebraic K-theory decomposes into eigenspaces of Adams operations:

$$\mathcal{K}_{p,X} = \bigoplus_{i=0}^{p} \mathcal{K}_{p,X}^{(i)}.$$

  - Milnor K-theory $\mathcal{K}_{p,X}^{\mathrm{M}}$ involves only pth eigenspace $\mathcal{K}_{p,X}^{(p)}$.
  - Example: $T\mathcal{K}_{3,X} \cong \Omega^2_{X/\mathbb{Q}} \oplus \mathcal{O}_X$ but $T\mathcal{K}_{3,X}^{\mathrm{M}} \cong \Omega^2_{X/\mathbb{Q}}$.

    Missing $\mathcal{O}_X$ summand!
  - Hence, Green-Griffiths tangent group $T_{\mathrm{GG}} \mathrm{Ch}_X^p$ neglects the other eigenspaces!
- $T_Y \mathrm{Ch}_X^p$ can "see" all the eigenspaces.
- Also allows arbitrary nilpotent structure (not just first order)!

Computing the Tangent Groups

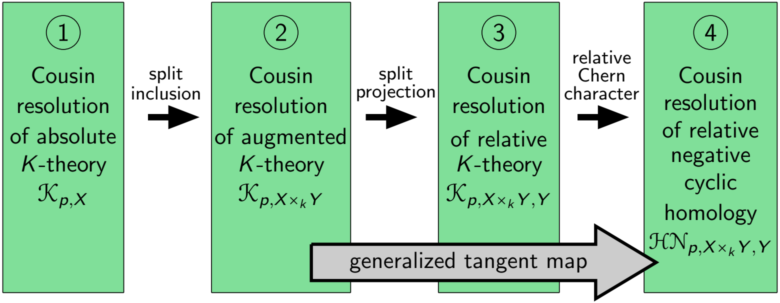
- Can we compute the groups $T_Y \mathrm{Ch}_X^p$? If so, how?
- Recall overarching goal: define computable invariants involving algebraic cycles, and show how to compute them!
- Computationally:
  - Cycle groups: **hard.**
  - Chow groups: **hard.**
  - Algebraic K-theory: **hard.**
  - Cyclic homology: **relatively easy.**
  - Differential forms: **easy.**
- Strategy: **use Chern character to convert "hard" K-theory into "easy" cyclic homology or differential forms.**
- Examples:

$$T\mathcal{K}_{p,X}^{\mathrm{M}} \cong \Omega_{X/\mathbb{Q}}^{p-1}.$$

$$\mathcal{K}_{p,X\times_k Y,Y} \cong \mathcal{HN}_{p,X\times_k Y,Y}.$$

Method: Coniveau Machine

- Coniveau machine converts information about $T_Y \mathrm{Ch}^p_X$ into information about negative cyclic homology.
- Schematic diagram:



- **Nontrivial:** The columns actually exist and fit together as shown!

Existence of Coniveau Machine: Early Versions

- Green-Griffiths [1]:
  - First-order theory, characteristic zero.
  - Focus on curves and surfaces.
  - Milnor K-theory, differential forms.
  - Define "tangent sequence to BGQR" for a surface.
  - Use different terminology.
- Sen Yang [19]:
  - First-order theory, characteristic zero.
  - Arbitrary dimensions
  - Also worked on typical curves (?)
  - Different terminology.
- Dribus-Hoffman-Yang upcoming [20].

Existence of the Machine: Background

General version (for Chow groups) requires:

- **CSS:** Grothendieck-Hartshorne circa 1960 [16].
- **Bloch's formula:** Bloch-Quillen, by 1972 [7], [9].
- **Effaceability of Bass-Thomason K-theory:** Colliot-Thélène, Hoobler, and Kahn 1997 [17]; implicit in Thomason 1990 [21].
- **Effaceability of negative cyclic homology:** implicit in Weibel 1991 [22] and Keller 1998 [23].
- **Relative algebraic Chern character an isomorphism:** (nilpotent case) Cortiñas et al. 2008 [15].

Column 1: Cousin Resolution of $\mathcal{K}_{p,X}$

- Standing assumptions:
  - **S** a "suitable" category of schemes over a field k.
  - X in **S** smooth algebraic variety.
  - Y in **S** fixed separated scheme.
- **To show:** this resolution exists!
- Easy: this is just the BGQR via the CSS:

$$0 \longrightarrow \mathcal{K}_{p,X} \longrightarrow \coprod_{x \in \mathrm{Zar}^0_X} \underline{K_p(k_x)} \longrightarrow \coprod_{x \in \mathrm{Zar}^1_X} \underline{K_{p-1}(k_x)} \longrightarrow \dots \dots \longrightarrow \coprod_{x \in \mathrm{Zar}^p_X} \underline{K_0(k_x)} \longrightarrow 0$$

- Requires Quillen's *devissage* to use residue fields k_x; applies only in smooth case!
- More general Bass-Thomason form using supports:

$$0 \longrightarrow \mathcal{K}_{p,X} \longrightarrow \coprod_{x \in \mathrm{Zar}^0_X} \underline{K_{p,X \text{ on } x}} \longrightarrow \coprod_{x \in \mathrm{Zar}^1_X} \underline{K_{p-1,X \text{ on } x}} \longrightarrow \dots \dots \longrightarrow \coprod_{x \in \mathrm{Zar}^p_X} \underline{K_{0,X \text{ on } x}} \longrightarrow 0$$

Columns 2 and 3: Multiplying by Y

- **To show:** Cousin resolutions of $\mathcal{K}_{p,X\times_k Y}$ and $\mathcal{K}_{p,X\times_k Y,Y}$ exist.
- Problem: scheme $X\times_k Y$ not smooth!
- Solution: define a new substratum:

$$X \mapsto \mathbf{K}_X^Y := \mathbf{K}_{X\times_k Y}.$$

  - Bass-Thomason form applies directly for $\mathcal{K}_{p,X}^Y = \mathcal{K}_{p,X\times_k Y}$:

$$0 \longrightarrow \mathcal{K}_{p,X}^Y \longrightarrow \coprod_{x\in \mathrm{Zar}_X^0} \underline{K_{p,X}^Y \text{ on } x} \longrightarrow \coprod_{x\in \mathrm{Zar}_X^1} \underline{K_{p-1,X}^Y \text{ on } x} \longrightarrow \cdots \cdots \longrightarrow \coprod_{x\in \mathrm{Zar}_X^p} \underline{K_{0,X}^Y \text{ on } x} \longrightarrow 0$$

  - Follows Colliot-Thélène, Hoobler, and Kahn's idea [17]: "new theories out of old."
- For $\mathcal{K}_{p,X\times_k Y,Y}$, take fiber $\mathbf{K}_X^Y \mapsto \mathbf{K}_X$; gives third column and maps for first three columns (details!)

Column 4: Relative Algebraic Chern Character

- **To show:**
  1. Cousin resolution of $\mathcal{HN}_{p,X\times_k Y,Y}$ exists.
  2. Cousin resolutions of $\mathcal{K}_{p,X\times_k Y,Y}$ and $\mathcal{HN}_{p,X\times_k Y,Y}$ are isomorphic as complexes.
- First part follows as for **K**, since **HN** defines effaceable CSS.
- Second part requires $X \mapsto X \times_k Y$ to be nilpotent.
- Then follows from fact that relative algebraic Chern character is isomorphism:

$$\mathrm{ch}_{n,X\times_k Y,Y} : \mathcal{K}_{p,X\times_k Y,Y} \to \mathcal{HN}_{p,X\times_k Y,Y}.$$

- **This completes the coniveau machine!**

Theorem and Corollary

- **Theorem (DHY, 2012):** The coniveau machine for Chow groups exists. That is, given a nilpotent augmentation $X \mapsto X \times_k Y$ of a smooth algebraic variety X over a field k, the $-p$th rows of the coniveau spectral sequences for absolute, augmented, and relative Bass-Thomason algebraic K-theory and negative cyclic homology sheafify to yield flasque resolutions of the corresponding sheaves on X, and the relative algebraic Chern character induces a functorial isomorphism between the resolutions of relative K-theory and relative negative cyclic homology.
- **Corollary:** The generalized tangent groups $T_Y\mathrm{Ch}^p_X$ of the Chow groups $T\mathrm{Ch}^p_X$ may be computed via negative cyclic homology in the nilpotent case:

$$T_Y\mathrm{Ch}^p_X \cong H^p_{\mathrm{Zar}}(X, \mathcal{HN}_{p, X\times_k Y, Y}).$$

Generalized Coniveau Machine

- Coniveau machine generalizes to "commutative diagram of functors and natural transformations with exact rows:"

$$\begin{array}{ccccc}
E_{H_{\mathrm{rel}},\mathbf{S}} & \xrightarrow{\;i\;} & E_{H_{\mathrm{aug}},\mathbf{S}} & \xrightarrow{\;j\;} & E_{H,\mathbf{S}} \\
\Big\downarrow{\scriptstyle \mathrm{ch}_{\mathrm{rel}}}\ \wr & & \Big\downarrow{\scriptstyle \mathrm{ch}_{\mathrm{aug}}} & & \Big\downarrow{\scriptstyle \mathrm{ch}} \\
E_{H^{+}_{\mathrm{rel}},\mathbf{S}} & \xrightarrow{\;i^{+}\;} & E_{H^{+}_{\mathrm{aug}},\mathbf{S}} & \xrightarrow{\;j^{+}\;} & E_{H^{+},\mathbf{S}}
\end{array}$$

- Here:
  - $\mathbf{S}$: category of spaces; e.g., schemes over a field k.
  - H: CTS on $\mathbf{S}$; e.g., algebraic K-theory.
  - H^{+}: "additive version" of H; e.g., negative cyclic homology.
  - $E_{H,\mathbf{S}}$ etc.: functors from spaces to CSS's.
  - ch etc.: "logarithmic-type transformations;" e.g., algebraic Chern character.

Future Directions I

- Compute new invariants:
  - $T\mathcal{K}_{3,X} \cong \Omega^2_{X/\mathbb{Q}} \oplus \mathcal{O}_X$; look in $\mathcal{O}_X$ piece!
  - Higher-order deformations.
- Symbolic K-theory:
  - My paper *A Goodwillie-type Theorem for Milnor K-Theory.*
  - Favorable remarks from Van der Kallen, Hesselholt, Stienstra.
  - Van der Kallen, Hesselholt: "generalize via de Rham Witt theory."
  - Stienstra: "combine with Cartier-Dieudonné theory."
  - Loday symbols: reach other Adams eigenspaces!
- Geometric interpretations à la Green-Griffiths.
  - "Formal" versus "geometric" objects.
  - Infinitesimal existence results.

Future Directions II

- Infinitesimal filtrations of Ch_X^p:
  - Related to Beilinson's conjectured filtrations of Ch_X^p.
  - Green-Griffiths version via differential forms.
  - Immediate reconsideration via cyclic homology.
- Abelian sums and residues.
- Positive characteristic. Much of our existing theory applies, but there are important new issues.
- Higher Chow groups.
- Other effaceable CTS's. Colliot-Thélène, Hoobler, and Kahn's idea [17] list a dozen; more have recently come to light.

THANKS!

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