Abelian Sums and Higher Tangents on Algebraic Curves

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1 Introduction

This paper represents an attempt to make sense of one of the many points of view [GG] introduce in the first two chapters of their book. Here abelian sums are used to give an equivalence relation on arcs of effective zero-cycles, and thus implicitly a definition of the corresponding tangent spaces to $X^{(m)}$ and $Z^1(X)^+$ (where X is a smooth algebraic curve over $\mathbb{C}$ and $Z^1(X)^+$ is the space of effective zero-cycles). This equivalence relation is then shown to be the same as that given by an independent but very sketchy heuristic argument. Intended only for motivation, this point of view is not developed systematically. Trying to understand it more rigorously raises some obvious questions:

1. Can the equivalence given by abelian sums be understood in more familiar or transparent terms?

2. Do we want to use this approach to give a *definition* of the tangent space, or relate it back to a more basic definition?

The answer to the first question (subsection 5.2) is that abelian sums can be viewed as (locally defined) functions on the product X^m which descend to the symmetric product $X^{(m)}$. These functions are simply linear combinations of sums of powers (in a local coordinate system). The equivalence given by abelian sums is thus seen to be analogous to the equivalence used to define tangent spaces in differential geometry, in which two arcs are equivalent if and only if the directional derivatives of all functions are the same for both arcs. However, abelian sums (seemingly) give a very restricted class of functions which might be expected to result in a coarse equivalence relation.

Moving on to the second question, we could define the tangent spaces to $X^{(m)}$ (and hence $Z^1(X)^+$) by means of a similar equivalence relation, except using *all* appropriate functions, instead of a restricted class. The meaning of appropriate depends on what category we are working in, but roughly speaking should refer to all functions on X^m which locally descend to $X^{(m)}$. This involves the canonical branched covering $X^m \to X^{(m)}$, and subsection 5.3 is a brief general discussion of equivalence of arcs in situations involving coverings and branched coverings. In subsection 5.4 I narrow the topic to symmetric products of algebraic curves and sketch the proof that the restricted class of abelian sums in fact gives the same equivalence as the class of all locally symmetric functions. This puts the proceedings on firmer ground.

Having addressed these preliminary questions, the next issue is:

3. How can we use these definitions and calculations to obtain formulas or other insight regarding the tangent spaces and higher-order jet spaces?

[GG] use the abelian sum approach to prove a residue formula which allows them to describe the tangent space (i.e. first-order equivalence) by means of a pairing between principal parts and holomorphic forms. In subsection 5.5, I discuss representing arcs of zero-cycles as arcs of divisors. In subsections 5.6 and 5.7 I give the generalization of the residue formula for higher-order equivalence. In subsection 5.8 and 5.9 I begin to discuss the implications for tangent spaces and jet spaces at effective zero cycles.

There remain issues to be resolved:

4. The object which [GG] identify as the tangent space does not resemble the intuitive idea of a tangent space, since it involves equivalence classes of arcs through many different points in $Z^1(X)^+$ (namely all multiples of a given point $x \in X$). I give (subsection 5.8) a definition of the tangent space at a single point, which is therefore somewhat different from their definition.

5. What about noneffective cycles? The heuristic discussion at the beginning of chapter 2 in [GG] involves the entire sheaf $\mathcal{Z}^1(X)$, but the treatment of abelian sums is limited to effective cycles. However, this treatment seems sufficient to give the results which generalize. This seems to imply that a great deal of structure is somehow being ignored, or else is not important in the curve case. Obviously any cycle is a difference of effective cycles, but difficulties arise when arcs of cycles are considered. For instance, how should creation/annihilation arcs be dealt with?

6. What is the topology on $Z^1(X)$? For instance, is the empty cycle near to any difference of nearby points on X because there is a creation/annihilation arc connecting them?

2 Functions defined by Integration and Abelian Sums

Let ω be a holomorphic 1-form defined on some simply connected open set $U \subset \mathbb{C}$ and $x_0 \in U$ an arbitrary lower limit of integration. By Cauchy's theorem, the integral $\int_{x_0}^{x} \omega$ is independent of path in U for any $x \in U$. Therefore, we have a well-defined function

$$I(-,\omega) : U \to \mathbb{C}$$
$$x \mapsto I(x,\omega) := \int_{x_0}^{x} \omega$$

Now suppose X is an algebraic curve over the complex numbers. Locally on X we have the same picture as before; every point has a simply connected neighborhood $U \subset X$ along with functions

$$U \to \mathbb{C}$$
$$x \mapsto I(x, \omega) := \int_{x_0}^{x} \omega$$

Definition 1 *The class of functions $U \to \mathbb{C}$ defined by integration of all $\omega \in \Omega^1_{X/\mathbb{C}}(U)$ with all possible lower limits of integration $x_0 \in X$ will be called $I(U)$. The subclass of $I(U)$ obtained from a fixed lower limit of integration x_0 will be called $I_{x_0}(U)$. The limit $lim_{x_0 \in U} I_{x_0}(U)$ will be called I_{x_0}.*

Fact 1 *I(U) consists of all holomorphic functions on U which vanish somewhere in U. Also $I_{x_0}(U) \subset m_{x_0} \subset \mathcal{O}_{X,x_0}$ and $I_{x_0} = m_{x_0}$.*

Proof: For the second statement, $\int_{x_0}^{x} \omega$ is holomorphic and $\int_{x_0}^{x} z^k dz = \frac{1}{k+1}(x^{k+1} - x_0^{k+1})$ vanishes at x_0. Extend by linearity to conclude the same for general ω. For the first statement, $I(U) = \cup_{x_0 \in X} I_{x_0}(U)$. For the third statement, if $f \in m_{x_0}$, then $df \in \Omega^1_{X/\mathbb{C},x_0}$ and $\int_{x_0}^{x} df = f(x)$.

□

Now let

$$x : \mathbb{C} \to U \subset X$$
$$t \mapsto x(t)$$

be an arc in X originating at some point $x(0) \in X$. Choose $x(0) = x_0$. We can form the composite function

$$\mathbb{C} \to \mathbb{C}$$
$$t \mapsto I(x(t), \omega) = \int_{x_0}^{x(t)} \omega$$

and take the directional derivative

$$I_1(\omega, (x(t))) := \frac{d}{dt} I(\omega, (x(t)))|_{t=0}$$

Since differentiation is local, we are taking the directional derivatives of all elements of $I_{x_0} = m_{x_0}$. Two arcs $x_1(t), x_2(t)$ are equivalent to first order at x_0 (i.e. define the same element of the tangent space $T_{x_0}X$) if and only if the corresponding directional derivatives are the same for all elements of $\mathcal{O}_{X,x_0}$, but the restricted class of functions I_{x_0} gives the same equivalence; indeed the only restriction is that the constant term (expressed in powers of $(x - x_0)$) is zero, and constants do not effect the directional derivatives. Thus, $x_1(t), x_2(t)$ are equivalent to first order if and only if

$$I_1(x_1(t), \omega) = I_1(x_2(t), \omega)$$

for all $\omega \in \Omega^1_{X,x_0}$.

Similarly, we may define functions from appropriate subsets of X^m into $\mathbb{C}$. For simplicity, we only consider product subsets $U^m = U \times ... \times U$, where U is a simply connected open set in X. We have functions

$$U^m \to \mathbb{C}$$
$$(x_1, ..., x_m) \to \sum_{i=1}^{m} \int_{x_{i0}}^{x_i} \omega_i$$

Likewise, we have functions from (analogous subsets of) $lim_{m\to\infty} X^m$ into $\mathbb{C}$ using the same formula.

Definition 2 *Denote the collections of such functions by $I^m(U)$ and $I^*(U)$, respectively.*

We want to know when elements of $I^m(U)$ or $I^*(U)$ (locally) descend to functions on $X^{(m)}$ or $Z^1(X)^+ := lim_{m\to\infty} X^{(m)}$ (the space of effective zero cycles).

Fact 2 $\sum_{i=1}^{m} \int_{x_{i0}}^{x_i} \omega_i$ *descends to $X^{(m)}$ or $Z^1(X)^+$ iff $\omega_i = \omega$ is the same for all i.*

Proof: $\sum_{i=1}^{m} \int_{x_{i0}}^{x_i} \omega_i$ is a power series, each term of which is a scalar multiple of a power of one of the x_i's. It is invariant under permutations of the variables if and only if the coefficient of x_i^k equals the coefficient of x_j^k for all i, j, k. But the x_i^k term comes from integrating the $x^{k-1}dx$ part of ω_i and the x_j^k term comes from integrating the $x^{k-1}dx$ part of ω_j. Thus $\omega_i = \omega_j$. Conversely if $\omega_i = \omega_j$, then the coefficient of x_i^k equals the coefficient of x_j^k for all i, j, k. (The constant part coming from the lower limits of integration is obviously invariant under permutations of the variables.)

□

Definition 3 *Let $I^{(m)}(U)$ be the subset of $I^m(U)$ consisting of functions which locally descend to $X^{(m)}$,and $I^{(*)}(U)$ the subset of $I^*(U)$ consisting of functions which locally descend to $Z^1(X)^+$. These functions are of the form*

$$U^{(m)} \to \mathbb{C}$$
$$x_1 + ... + x_m \to \sum_{i=1}^{m} \int_{x_{i0}}^{x_i} \omega$$

for $x_1, ..., x_m \in U$ and $\omega \in \Omega^1_{X/\mathbb{C}}(U)$. The object

$$\int_{x_{i0}}^{x_i} \omega$$

is called an abelian sum.

Discussion: While discussing equivalence of arcs in X, we were able to make the situation completely local by taking the lower limit of integration x_0 to be the same as the initial point $x(0)$ of the arcs under consideration. This allowed us to work with forms in the stalk $\Omega^1_{X/\mathbb{C},x_0}$. This approach enables us to describe the tangent space $T_{x_0}X$ in terms of linear functionals on $\Omega^1_{X/\mathbb{C},x_0}$. In the present case, it is necessary for our forms ω to be defined in a common neighborhood of the points $x_1, ..., x_m$. We must also consider the lower limits of integration. [GG] use a uniform limit of integration $x_{i0} = x_0$ for all i in ch. 2 of their book, but they are only considering the case of a multiple point mx. For our purposes, it is better to take $x_{i0} = x_i(0)$, allowing the lower limits of integration to coincide with the initial points of the arcs as in the case of arcs in X. This allows us to work with forms in $\cap_{i=1}^m \Omega^1_{X/\mathbb{C},x_{i0}}$.

Since any holomorphic form ω can be written in terms of the coordinate system on U as a linear combination of powers (times dx), any abelian sum can be written as a linear combination of sums of powers $\sum_{i=1}^m x_i^k$. In the converse direction, we will only need the fact that, up to constants, any linear combination of sums of powers which is a polynomial (i.e. not an infinite series) can be written as an abelian sum.

We may use abelian sums to investigate equivalence of arcs in $X^{(m)}$ and $Z^1(X)^+$. Explicitly, given an arc $\sum_{i=1}^m x_i(t)$ in $X^{(m)}$ we can form the composite map

$$t \mapsto z(t) \mapsto I(z(t), \omega) := \sum_{i=1}^m \int_{x_{i0}}^{x_i(t)} \omega$$

and take directional derivatives.

Definition 4 *The quantity*

$$I_n(z(t), \omega) := \frac{1}{n!} \frac{d^n}{dt^n} (\sum_{i=1}^m \int_{x_{i0}}^{x_i(t)} \omega)_{t=0}$$

is called the nth abelian invariant of $z(t)$ with respect to ω.

Note that this definition does not depend on the choices of lower limits of integration, which become irrelevant after differentiating.

This definition allows us to define equivalence classes of arcs originating at $\sum_{i=1}^m x_i(0)$ in $X^{(m)}$ or $Z^1(X)^+$: two arcs are equivalent up to order n if all their abelian invariants are the same up to order n. But why should we be interested in this equivalence? As we saw above, $I^{(m)}$ contains only linear combination of sums of powers, a seemingly small class of functions. Thus we might expect a coarse equivalence.

Taking a step back, we have not yet even defined the tangent spaces (and higher order analogues) to $X^{(m)}$ and $Z^1(X)^+$, so we have not yet specified what equivalence of arcs should mean. In the next section, we give a natural way to define equivalence of arcs in $X^{(m)}$ and $Z^1(X)^+$, using the canonical branched coverings $X^m \to X^{(m)}$. We then prove that $I^{(m)}$ and $I^{(*)}$ give this same equivalence.

3 Arc spaces, covering spaces, branched coverings, symmetric products

We want to express a tangent space as a space of arcs modulo some equivalence relation (given by pairing arcs against certain functions or forms).

Let (X, x) and (Y, y) be pointed sets. The set of all pointed maps $(Y, y) \to (X, x)$ will be called the (Y, y)-*arc space of* X *at* x. It is denoted $A((Y, y), (X, x))$. For us, Y will be a curve and X a symmetric product or space of cycles. We also restrict the class of arcs by requiring other properties (such as smoothness) as needed.

We are interested in equivalence classes of arcs. If X has a differentiable structure, then we can obtain an equivalence relation as in differential geometry by pairing arcs against differentiable functions or forms. There are questions concerning the "right" way to define equivalence in this context, as [GG] discuss situations in which higher forms give "new information."

We may investigate arc spaces on X by means of coverings or branched coverings of X by manifolds X'. If $X' \to X$ is a covering, then we can transfer everything to a sheet of X'; the local isomorphism between a neighborhood of x and its preimages induces an isomorphism of arc spaces. If $X' \to X$ is a branched covering and the point $x \in X$ is a branch point, then the situation is more complicated; an arc in X may have multiple liftings originating from any given lifting x' of x in X'. Moreover, some of these liftings may be smooth in X' while others are not. Indeed, when considering how to restrict the class of arcs in X when $X' \to X$ is branched, we must define what is meant by smooth, etc. I will adopt the convention that γ is smooth in X if it has a smooth lifting in X'.

A glaring problem with this approach is that it is (seemingly, at least) not intrinsic to X; there is no guarantee that another branched covering $X'' \to X$ gives the same answer. This may not be a problem if we have a canonical covering with appropriate universal properties.

Roughly speaking, we want to lift arcs to X' to test them for equivalence. If we define equivalence by pairing arcs against certain functions or forms, then we may either use all the functions or forms (in a given class) on X', or only those which locally descend to X. These two approaches do not give the same answer. Using only functions or forms which locally descend makes more sense formally, but does not necessarily give geometrically intuitive results.

The branched covering we are particularly interested in is the canonical map from the product to the symmetric product

$$X^m \to X^{(m)}$$

$$(x_1, ..., x_m) \mapsto x_1 + ... + x_m$$

where X is a manifold (usually a smooth complex variety). The branch locus consists of the "diagonals" where two or more points x_i, x_j are equal.

4 Locally symmetric functions and the definition of equivalence of arcs

Equivalence of two arcs γ_1, γ_2 in $X^{(M)}$ is measured by equivalence of their liftings with respect to differentiable functions on X^M which locally descend to $X^{(M)}$; such functions are called locally symmetric functions.

Definition 5 *Let $a = (a_{11}, ..., a_{1n_1}, ..., a_{m1}, ..., a_{mn_m}) \in \mathbb{C}^M$, where $M := \sum_{i=1}^m n_i$, and where $a_i = a_{i1} = ... = a_{in_i}$ are the distinct values. A function $f : \mathbb{C}^M \to \mathbb{C}$ defined near a is called locally symmetric at a if it is symmetric in each set of variables $(x_{11}, ..., x_{1n_1}), ..., (x_{m1}, ..., x_{mn_m})$ in some neighborhood of a. In other words, f is locally symmetric if it is invariant under every permutation that fixes a.*

Locally symmetric functions at a are functions whose asymmetry (if any) is not detectable near a in the following sense: $X^{(M)}$ is the quotient of X^M by the symmetric group S_M in the obvious way. Functions are symmetric if they are invariant under the action of the symmetric group, but since this action in general carries points outside of a small neighborhood, symmetry is not a local property (at least in the differentiable case). However, if a is on a diagonal, there is a nontrivial subgroup of S_M, which we will denote S_a and call the *locally symmetric group at a*, which fixes a. Therefore, invariance of functions under the action of S_a is a local property, which we call local symmetry. This level of detail is perhaps unecessary in the algebraic case, where open sets are large.

Definition 6 *Two arcs γ_1, γ_2 originating at a in $X^{(M)}$ or $Z^1(X)^+$ are equivalent to order n if*

$$\frac{d^k}{dt^k} f(\gamma_1(t))|_{t=0} = \frac{d^k}{dt^k} f(\gamma_2(t))|_{t=0}$$

for all $1 \le k \le n$ and all locally symmetric f.

Theorem 1 *$I^{(M)}$ distinguishes equivalence of arcs in $X^{(M)}$. $I^{(*)}$ distinguishes equivalence of arcs in $Z^1(X)^+$.*

Proof: (Sketch)A tedious lemma (which we omit) shows that sums of powers give the same equivalence as local sums of powers. Elementary locally symmetric functions can be written in terms of local sums of powers, and locally symmetric functions can be written in terms of elementary locally symmetric functions.

□

With this as motivation, we make the following definition:

Definition 7 *Two arcs γ_1, γ_2 in $X^{(M)}$ or $Z^1(X)^+$ are equivalent to order n if*

$$\frac{d^k}{dt^k} I(\gamma_2(t), \omega)_{t=0} = \frac{d^k}{dt^k} I(\gamma_2(t), \omega)_{t=0}$$

for all $1 \le k \le n$ and all holomorphic ω.

5 Abelian Sums and Arcs of Divisors

It will be useful to express zero-cycles as divisors of rational functions, then work with these rational functions to obtain results. Abel's theorem tells when this can be done globally, but in the present case we only need local expressions, and these are easily obtained.

Consider an arc $z(t) = \sum_i x_i(t)$ of effective zero cycles given by a regular mapping from an algebraic curve B with local uniformizing parameter t into $X^{(M)} \subset Z^1(X)^+$. Let x be a local holomorphic coordinate on X. Consider the function

$$f(x,t) = \prod_{i=1}^{m}(x - x_i(t))$$

For any fixed t, f is a rational function, in fact a polynomial, in the local coordinate x. Therefore we may think of $f(x,t)$ as an arc of rational functions; i.e. a one-parameter map into the space of rational functions. Locally $z(t)$ can be expressed as

$$z(t) = div(f(t))$$

Since our use of f will be completely local (specifically, taking residues) we do not need to worry about global issues (e.g. poles of f outside the local coordinate system).

In general, the $x_i(t)$ will be Puiseaux series in t, but the regularity of the map $B \to X^{(M)}$ insures that the coefficients of powers of x in $f(x,t) = \prod_{i=1}^{m}(x - x_i(t))$, which are the elementary symmetric functions in the $x_i(t)$, involve no fractional powers. Therefore we may express $f(x,t)$ as

$$= f(x,t) = f_0(x) + f_1(x)t + f_2(x)t^2 + ...$$

where the $f_i(x)$ are polynomials.

For a holomorphic 1-form ω on X, we have discussed the *abelian sum*

$$I(t) := \sum_i \int_{x_{i0}}^{x_i(t)} \omega$$

and the *abelian invariants* $I_n(z(t),\omega)$ given by

$$I_n(z(t),\omega) := \frac{1}{n}\frac{d^n}{dt^n}(\sum_i \int_{x_{i0}}^{x_i(t)} \omega)_{t=0}$$

In the case where $x_{i0} = x_0$ is the same for all i, [GG] prove that

$$I_1(z(t),\omega) = -Res_{x_0}\frac{f_1}{f_0}\omega$$

(using my notation; their proof is on p 25-26 of [GG]).

In the next few sections, we will develop and prove similar results in the general case (i.e. without the assumption that $x_{i0} = x_0$ for all i) for the higher abelian invariants I_n.

6 Example: A single moving point given by a power series

We will first consider the case of a single moving point on X whose position is given by a power series. This case suggests many features of the general results to follow. Explicitly, we will consider the following situation:

Let X be a Riemann surface and $x(0) \in X$ a point.

Let x be a local coordinate on X near $x(0)$. Centering the local coordinate at $x(0)$ would simplify our expressions by making certain terms vanish, but for illustrative purposes it is better to keep these terms.

Let $x(t) = a_0 + a_1 t + a_2 t^2 + ...$ be the power series expansion giving the position of $x(t)$ at time t in terms of x. Locally we have a holomorphic map:

$$\mathbb{C} \to \mathbb{C}$$
$$t \mapsto (x = x(t) = a_0 + a_1 t + a_2 t^2 + ...)$$

Observe that $x(0) = a_0$.

Let $\omega = x^k dx$ be a holomorphic 1-form on X (locally), where k is a positive integer. In general we will be considering $\mathbb{C}$-linear combinations of such forms for different values of k.

Let x_0 (NOT the same as $x(0)$ in this example) be the lower limit of integration. Consider the abelian sum

$$I(t) = \int_{x=x_0}^{x(t)} \omega = \frac{1}{k+1}(x(t)^{k+1} - {x_0}^{k+1})$$

The abelian invariant $I_1(x(t), \omega)$ is given by

$$I_1(x(t), \omega) = \frac{d}{dt}(\frac{1}{k+1}(x(t)^{k+1} - {x_0}^{k+1}))_{t=0} = (x(t)^k x'(t))_{t=0} = x(0)^k x'(0) = {a_0}^k a_1$$

The second-order abelian invariant $I_2(x(t), \omega)$ is given by

$$I_2(x(t), \omega) := \frac{1}{2!}\frac{d^2}{dt^2}(\int_{x_0}^{x(t)} \omega)_{t=0}$$

$$= \frac{1}{2}\frac{d}{dt}(x(t)^k x'(t))_{t=0} = \frac{1}{2}(x(0)^k x''(0) + kx(0)^{k-1}(x'(0))^2) = \frac{1}{2}(2{a_0}^k a_2 + k{a_0}^{k-1}{a_1}^2)$$

Similarly, we can compute

$$I_3(x(t),\omega) = \frac{1}{6}(6{a_0}^k a_3 + 6k{a_0}^{k-1}a_1a_2 + k(k-1){a_0}^{k-2}{a_1}^3)$$

$$I_4(x(t),\omega) = \frac{1}{24}(24{a_0}^k a_4 + 24k{a_0}^{k-1}a_1a_3 + 12k{a_0}^{k-1}{a_2}^2$$
$$+12k(k-1){a_0}^{k-2}{a_1}^2a_2 + k(k-1)(k-2){a_0}^{k-3}{a_1}^4)$$

The reason for carrying these calculations so far is to illustrate a connection between the above expressions and the power series expansion of the natural logarithm function, described below.

I will now reformulate these results in terms of residues.

We want to be able to describe the abelian invariants I_n arising from an arc of zero cycles $div(f_0 + tf_1 + t^2f_2 + ...)$ in terms of the functions f_0, f_1, f_2, etc. In the present case, the arc of zero cycles is simply the "moving singleton"

$$x(t) = div(x - x(t)) = div(x - a_0 - a_1t - a_2t^2 - ...)$$

In other words, in our case $f_0 = x - a_0$, $f_1 = -a_1$, $f_2 = -a_2$, etc.

Remark: For a given arc $z(t) = div(f(t))$, the "arc of rational functions" $f(t)$ is only determined up to units in the local ring $\mathcal{O}_{X,x}$; i.e. $div(f(t)) = div(u(t)f(t))$ for $u(t)$ a unit. Above, I *assumed* that $f(t) = \prod_{i=1}^m (x - x_i(t))$, but we could start with an arbitrary $f(t) = u(t)\prod_{i=1}^m (x - x_i(t))$. This has no effect on [GG]'s residue formula for I_1 because the expression they obtain, namely $-Res_x(\frac{f_1\omega}{f_0})$, has the same total exponent for the f_i's in the numerator as it does in the denominator, so that the contribution of multiplying by a unit cancels; the unit scales all of the f_i's in the same way. For the same reason, my formulas below are not affected by this ambiguity.

I will use the following elementary fact about computing residues:

Fact 3 *If $g(x)$ has a pole of order n at $x(0)$, then*

$$Res_{x(0)}g(x)dx = \frac{1}{(n-1)!}\lim_{x\to x(0)}\frac{d^{n-1}}{dx^{n-1}}(x - x(0)^n g(x)$$

We may now calculate the abelian invariants I_n in the case of a single moving point given by a power series with constant coefficients. Working backward,

$$I_1(x(t), x^k dx) = {a_0}^k a_1 = -\lim_{x\to a_0}(x - a_0)\frac{-a_1x^k}{x - a_0} =: -Res_{a_0}(\frac{-a_1x^k dx}{x - a_0}) = -Res_{x(0)}(\frac{f_1\omega}{f_0})$$

This is the result given on page 25 of [GG] in the case of an arc originating from a multiple point mx (though they mistakenly omit the minus sign).

Moving on to I_2, we want to write

$$I_2(x(t),\omega) = \frac{1}{2}(2{a_0}^k a_2 + k{a_0}^{k-1}{a_1}^2)$$

in terms of residues. The first term is $-Res_{x(0)}(\frac{f_2\omega}{f_0})$ by similar reasoning to the case of I_1. The second term is

$$\frac{1}{2}k{a_0}^{k-1}{a_1}^2 = \frac{1}{2}\lim_{x\to a_0}(\frac{d}{dx}(x-a_0)^2\frac{x^k dx {a_1}^2}{(x-a_0)^2}) = \frac{1}{2}Res_{x(0)}(\frac{{f_1}^2\omega}{{f_0}^2})$$

.

Putting the two terms together yields the expression

$$I_2(x(t),\omega) = -Res_{x(0)}(\frac{f_2}{f_0} - \frac{1}{2}(\frac{f_1}{f_0})^2)\omega$$

Similarly, we obtain

$$I_3(x(t),\omega) = -Res_{x(0)}(\frac{f_3}{f_0} - \frac{f_1}{f_0}\frac{f_2}{f_0} + \frac{1}{3}(\frac{f_1}{f_0})^3)\omega$$

$$I_4(x(t),\omega) = -Res_{x(0)}(\frac{f_4}{f_0} - \frac{f_1}{f_0}\frac{f_3}{f_0} - \frac{1}{2}(\frac{f_2}{f_0})^2 + (\frac{f_1}{f_0})^2\frac{f_2}{f_0} - \frac{1}{4}(\frac{f_1}{f_0})^4)\omega$$

Consider the expressions

$$A_1 := \frac{f_1}{f_0}$$

$$A_2 := \frac{f_2}{f_0} - \frac{1}{2}(\frac{f_1}{f_0})^2$$

$$A_3 := \frac{f_3}{f_0} - \frac{f_1}{f_0}\frac{f_2}{f_0} + \frac{1}{3}(\frac{f_1}{f_0})^3$$

$$A_4 := \frac{f_4}{f_0} - \frac{f_1}{f_0}\frac{f_3}{f_0} - \frac{1}{2}(\frac{f_2}{f_0})^2 + (\frac{f_1}{f_0})^2\frac{f_2}{f_0} - \frac{1}{4}(\frac{f_1}{f_0})^4$$

What is interesting is that these expressions are the coefficients of the first four powers of t in the formal power series expansion for $ln(1+\alpha)$ with $\alpha := \frac{f_1}{f_0}t + \frac{f_2}{f_0}t^2 + ...$; i.e.

$$ln(1 + \frac{f_1}{f_0}t + \frac{f_2}{f_0}t^2 + ...) = A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4 + ...$$

This pattern continues, as shown in the next section. The higher-order abelian invariants I_n can be calculated in a similar way, but it is better to consider all the I_n's together, as explained below.

We can write:

$$I(z(t),\omega) = I_0 + I_1 t + I_2 t^2 + ...$$

(Taylor expansion near $t = 0$)

Since the lower limit of integration x_0 is arbitrarily chosen, I is determined only up to a constant; i.e. the number $I_0 = \sum \int_{x_0}^{x_i(0)} \omega$ depends on x_0. Therefore, I simply carry it as a constant through the rest of the calculation. Assuming that

$$I_n(z(t),\omega) = -Res_{x(0)} A_n \omega$$

as suggested above, we may write

$$I(z(t),\omega) = I_0 - Res_{x(0)} A_1 \omega t - Res_{x(0)} A_2 \omega t^2 - ...$$

$$= I_0 - Res_{x(0)}(A_1 t + A_2 t^2 + ...)\omega = I_0 - Res_{x(0)} ln(1 + \frac{f_1}{f_0}t + \frac{f_2}{f_0}t^2 + ...)\omega$$

But $1 + \frac{f_1}{f_0}t + \frac{f_2}{f_0}t^2 + ... = \frac{f(t)}{f_0}$, so finally

$$I(z(t),\omega) = I_0 - Res_{x(0)}(ln(f(t) - ln(f_0))\omega$$

This formula is proved in the next section. The steps just given are all reversible, so we may conclude that

$$I_n(z(t),\omega) = -Res_{x(0)} A_n \omega$$

as suggested by the low-order examples above.

7 The Residue Formula for $I(z(t),\omega)$

We begin with the case of arcs originating from a multiple point mx. Then we will move on to the general case of arcs originating from an arbitrary point $\sum_{i=1}^{m} n_i x_i \in Z^1(X)^+$.

Let

$$z(t) = \sum_{i=1}^{m} x_i(t)$$

be an arc of zero cycles with $x_i(0) = x(0)$ for all i. Let x be a local coordinate centered at $x(0)$; i.e. $x_i(0) = 0$ for all i in terms of x. Write

$$z(t) = div(f(t)) = div(f_0 + f_1 t + f_2 t^2 + ...)$$

as in subsection 5.5. Note that $f(t)$ is determined only up to units in the local ring at $x = 0$; up to such we can assume

$$f(t) = \prod_{i=1}^{m}(x - x_i(t))$$

With this assumption, $f_0 = x^m$.

Let $x_0 = x(0) = 0$ be the lower limit of integration.

We are considering abelian sums

$$I(z(t), \omega) := \sum_{i=1}^{m} \int_0^{x_i(t)} \omega$$

for holomorphic forms ω.

We may expand $I(t) = I(z(t), \omega)$ as a Taylor series

$$I(t) = I_0 + I_1 t + I_2 t^2 + ...$$

where $I_0 = I(0) = \sum_{i=1}^{m} \int_{x_0}^{x}(0)\omega = 0$ because of the choice of lower limit of integration and the I_n's are the abelian invariants.

Proposition 1 $I(z(t), \omega) = -Res_{x(0)}(\ln(f(t)) - \ln(f_0))\omega$

Proof: (Hoffman)

Without loss of generality, we may assume that $\omega = x^k dx$ for some k.

Then

$$I(t) = \frac{1}{k+1} \sum_{i=1}^{m} x_i(t)^{k+1}$$

Now

$$f(t) = \prod_{i=1}^{m}(x - x_i(t)) = x^m \prod_{i=1}^{m}(1 - \frac{x_i(t)}{x}) = f_0 \prod_{i=1}^{m}(1 - \frac{x_i(t)}{x})$$

Take logarithms:

$$\ln(f(t)) = \ln(f_0) + \sum_{i=1}^{m} \ln(1 - \frac{x_i(t)}{x})$$

Apply the series for $\ln(1+\alpha)$ to the RHS:

$$\ln(f(t)) = \ln(f_0) - \sum_{i=1}^{m}(\sum_{l\geq 1}\frac{1}{l}(\frac{x_i(t)}{x})^l)$$

Now interchange the order of summation:

$$\ln(f(t)) = \ln(f_0) - \sum_{l\geq 1}\frac{1}{lx^l}(\sum_{i=1}^{m} x_i(t)^l)$$

Rearrange, multiply by $\omega = x^k dx$ and take residues at 0:

$$Res_0(\sum_{l\geq 1}\frac{1}{lx^l}(\sum_{i=1}^{m} x_i(t)^l))x^k dx = -Res_0(\ln(f(t)) - \ln(f_0))x^k dx$$

On the LHS, only the $l = k+1$ term contributes, yielding $\frac{1}{k+1}\sum_{i=1}^{m} x_i(t)^{k+1} = I(z(t), x^k dx)$. So we obtain

$$I(z(t), x^k dx) = Res_{x(0)}(\ln(f(t)) - \ln(f_0))x^k dx$$

completing the proof.

□

Now we generalize to the case where $z(t)$ originates from a point $\sum_{i=1}^{m} n_i x_i$. In this case

$$z(t) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} x_{ij}(t)$$

where $x_{ij}(0) = x_i$.

We may write $z(t) = div(f(t))$ where $f(t) = f_1(t)...f_n(t)$, where the $f_i(t)$ represent the various initial points in the sense that $f_i(t) = (x - x_{i1}(t))...(x - x_{in_i}(t))$ with $x_{ij}(0) = x_i$ for all $j = 1, ..., n_i$, and $x_i \neq x_j$ for $i \neq j$.

Let the x_i be the lower limits of integration. With these choices

$$I(z(t), \omega) = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\int_{x_i}^{x_{ij}(t)} \omega$$

In particular:

$$I(z(t), x^k dx) = \frac{1}{k+1} \sum_{i=1}^{m} \sum_{j=1}^{n_i} x_{ij}(t)^{k+1}$$

Rewrite $f_i(t)$ as

$$\begin{aligned} f_i(t) &= [(x - x_i) - (x_{i1}(t) - x_i)]...[(x - x_i) - (x_{in_i}(t) - x_i)] \\ &= (x - x_i)^{n_i}(1 - \frac{x_{i1}(t) - x_i}{x - x_i})...(1 - \frac{x_{in_i}(t) - x_i}{x - x_i}) \end{aligned}$$

Express $f(t)$ as

$$f(t) = (x - x_i)^{n_i}(1 - \frac{x_{i1}(t) - x_i}{x - x_i})...(1 - \frac{x_{in_i}(t) - x_i}{x - x_i}) \prod_{j \neq i} f_j(t)$$

Note that $(x - x_i)^{n_i} = f_i(0)$.

Take logarithms:

$$log(f(t)) = log(f_i(0)) + \sum_{j=1}^{n_i} log(1 - \frac{x_{ij}(t) - x_i}{x - x_i}) + \sum_{j \neq i} log(f_j(t))$$

Note that $log(f(t)) - log(f_i(0)) - \sum_{j \neq i} log(f_j(t)) = log(\frac{f(t)}{f_i(0) \prod_{j \neq i} f_j(t)}) = log(f_i(t)) - log(f_i(0))$. We get

$$log(f_i(t)) - log(f_i(0)) = \sum_{j=1}^{n_i} log(1 - \frac{x_{ij}(t) - x_i}{x - x_i})$$

Use the series for $log(1 + \alpha)$ on the RHS. We obtain:

$$log(f_i(t)) - log(f_i(0)) = -\sum_{j=1}^{n_i} \sum_{l=1}^{\infty} \frac{1}{l} (\frac{x_{ij}(t) - x_i}{x - x_i})^l$$

Rearrange, multiply by $x^k dx$, and take residues at x_i:

$$Res_{x_i}(\sum_{j=1}^{n_i} \sum_{l=1}^{\infty} \frac{1}{l} (\frac{x_{ij}(t) - x_i}{x - x_i})^l) x^k dx = -Res_{x_i}(log(f_i(t)) - log(f_i(0))) x^k dx$$

Since we are taking residues at x_i, express x^k in powers of $(x - x_i)$:

$$x^k = \sum_{p=0}^{k} \binom{k}{p} x_i^p (x - x_i)^{k-p}$$

The LHS becomes

$$Res_{x_i}(\sum_{j=1}^{n_i} \sum_{l=1}^{\infty} \frac{1}{l} (\frac{x_{ij}(t) - x_i}{x - x_i})^l) \sum_{p=0}^{k} \binom{k}{p} x_i^p (x - x_i)^{k-p} dx$$

We have terms

$$\frac{1}{l} (\frac{x_{ij}(t) - x_i}{x - x_i})^l) \binom{k}{p} x_i^p (x - x_i)^{k-p}$$

We have a nonzero residue iff $l = k - p + 1$. Now

$$\frac{1}{k - p + 1} \binom{k}{p} = \frac{1}{k+1} \binom{k+1}{p}$$

so the residue coming from this term is $\frac{1}{k+1}\binom{k+1}{p} x_i^p (x_{ij}(t) - x_i)^{k+1-p}$

Now $l \geq 1$, so p runs from 0 to k. Fixing j, we get

$$\sum_{p=0}^{k} \frac{1}{k+1} \binom{k+1}{p} x_i^p (x_{ij}(t) - x_i)^{k+1-p} = \frac{1}{k+1} (x_{ij}^{k+1} - x_i^{k+1})$$

Letting j vary, we get

$$\frac{1}{k+1} \sum_{j=1}^{n_i} (x_{ij}^{k+1} - x_i^{k+1}) = -Res_{x_i}(log(f_i(t)) - log(f_i(0))) x^k dx$$

Since this is true for all i, we get

$$\frac{1}{k+1} \sum_{i=1}^{m} \sum_{j=1}^{m_i} (x_{ij}^{k+1} - x_i^{k+1}) = -\sum_{i=1}^{n} Res_{x_i}(log(f_i(t)) - log(f_i(0))) x^k dx$$

The LHS is $I(t)$, and we can replace $x^k dx$ with a linear combination of such terms, so finally

Theorem 2 *Let $z(t) = div(f(t))$, where $f(t) = f_1(t)...f_n(t)$ and $f_i(t) = (x - x_{i1}(t))...(x - x_{im_i}(t))$ with $x_{ij}(0) = x_i$ for all $j = 1, ..., n_i$, and $x_i \neq x_j$ for $i \neq j$. Then*

$$I(z(t), \omega) = -\sum_{i=1}^{m} Res_{x_i}(log(f_i(t)) - log(f_i(0)))\omega$$

Proof: Foregoing discussion.

□

This formula encodes all the information from the abelian invariants I_n. It is to be understood in terms of the expansion of the right hand side according to the power series for $ln(1+\alpha)$. If

$$f_i(t) = f_{i0} + f_{i1}t + f_{i2}t^2 + ...$$

and

$$log(\frac{f_i(t)}{f_0}) = A_1 t + A_2 t^2 + ...$$

then the formula becomes

$$I(z(t), \omega) = -\sum_{j=1}^{\infty}(\sum_{i=1}^{m} Res_{a_i} A_{ij}\omega)t^j$$

and

$$I_n(z(t), \omega) = -\sum_{i=1}^{m} Res_{a_i} A_{ik}\omega$$

8 Abelian invariants and pullbacks of differential forms

Let

$$x : D \to X$$

$$t \mapsto x(t) = a_0 + a_1 t + a_2 t^2 + ...$$

be a moving point on X with multiplicity one. Let $\omega = x^k dx$ be a differential form on X. Calculate the pullback of ω to D. At this point we are only concerned with ordinary differentials, not absolute differentials. We get

$$x^*(\omega) = (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ...)^k d(a_0 + a_1 t + a_2 t^2 + a_3 t^3 ...)$$

$$= (a_0^k + k a_0^{k-1} a_1 t + (k a_0^{k-1} a_2 + \binom{k}{2} a_0^{k-2} a_1^2)t^2 + ...)(a_1 dt + 2a_2 t dt + 3a_3 t^2 dt + ...)$$

$$= a_0^k a_1 dt + (2a_0^k a_2 + k a_0^{k-1} a_1^2)t dt + (3a_0^k a_3 + 3k a_0^{k-1} a_1 a_1 + \binom{k}{2} a_0^{k-2} a_1^3)t^2 dt + ...$$

$$= I_1(x(t),\omega)dt + 2I_2(x(t),\omega)tdt + 3I_3(x(t),\omega)t^2dt + ...$$

This example suggests the following lemma:

Lemma 1 *Let $x : D \to X$ be an arc of zero cycles and $\omega \in \Omega^1_{X/\mathbb{C}}$. Then*

$$x^*(\omega) = \sum_{k=1}^{\infty} I_k(x(t),\omega)t^{k-1}dt$$

Proof: (Sketch)

$$I(x(t),\omega) = \int_{x(0)}^{x(t)} \omega(x') = \int_0^t x^*(\omega)(t') = \sum_{k=1}^{\infty} I_k(x(t),\omega)t^k$$

Now differentiate the last equality with respect to t.

Remark: the notation $\omega(x')$ indicates that ω is written in terms of the dummy variable x', and similarly for the pullback $x^*(\omega)(t')$, which is written in terms of the dummy variable t'.

□

Remark: I don't know how general this lemma is; it may work only for a single moving point, or only for effective cycles, etc.

Remark: Had we chosen to include absolute differentials, we would have written

$$d(a_0 + a_1t + a_2t^2 + a_3t^3 + ...) = da_0 + da_1t + a_1dt + da_2t^2 + 2a_2tdt + da_3t^3 + 3a_3t^2dt + ...$$

In the case of curves, when we contract with $\frac{\partial}{\partial t}$, the only terms that remain are those where we have differentiated with respect to t. Therefore, the absolute differentials do not come into the picture. For surfaces and other higher-dimensional varieties, we end up wedging together forms before applying $\frac{\partial}{\partial t}$, so the absolute differentials contribute.

9 BGQ/Cousin perspective

9.1 First-order case

The divisor sequence

$$0 \longrightarrow \mathcal{O}_X^* \xrightarrow{i} \mathcal{R}_X^* \xrightarrow{div} \underline{Z}^1(X) \longrightarrow 0$$

where the map i is inclusion and the map div sends a nonzero rational function to its divisor, may be rewritten

$$0 \longrightarrow K_1(\mathcal{O}_X) \xrightarrow{i} \underline{K}_1(\mathbb{C}(X)) \xrightarrow{div} \bigoplus_{y \in V^1(X)} \underline{K}_0(\mathbb{C}(y)) \longrightarrow 0$$

and is thus a special case of the Bloch-Gersten-Quillen (BGQ) sequence.

Meanwhile the principal parts sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{j} \mathcal{R}_X \xrightarrow{C_1} \mathcal{P}_X \longrightarrow 0$$

where the map j is inclusion and the map C_1 sends a rational function to its principal part, may be rewritten

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{j} \underline{H}^0_x(\mathcal{O}_X) \xrightarrow{C_1} \bigoplus_{y \in V^1(X)} \underline{H}^1_y(\mathcal{O}_X) \longrightarrow 0$$

where x is the generic point of X and $y \in X$ are the closed points. Elements of $H^1_y(\mathcal{O}_X$ are represented by symbols $[f^l, g]$, where f is a local defining equation for y; i.e. we have the (resolution?)

$$0 \longrightarrow F_1 \xrightarrow{f^l} F_0 \longrightarrow \mathcal{O}_X/f^l\mathcal{O}_X \longrightarrow 0$$

and $g \in \frac{Hom(F_1,\mathcal{O}_X)}{Hom(F_0,\mathcal{O}_X)}$

where we use the prescription

$$H^1_y(\mathcal{O}_X) = lim_l \frac{Hom(F_1,\mathcal{O}_X)}{Hom(F_0,\mathcal{O}_X)}$$

in [GG] on page 105 (very rough; iron out the details)

The map C_1 is defined as follows: given

$$g = \frac{h}{g_1^{l_1} \dots g_k^{l_k}} \in \mathbb{C}(X)$$

where $g_1, \dots, g_k, h$ are relatively prime and the g_i are irreducible, define

$$C_1(g) = \sum_{i=1}^{k} \frac{h}{g_1^{l_1} \dots \hat{g_i^{l_i}} \dots g_k^{l_k}} \in \bigoplus_y H^1_y(\mathcal{O}_X)$$

We then have the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & A(K_1(\mathcal{O}_X)) & \longrightarrow & A(\underline{K}_1(\mathbb{C}(X))) & \longrightarrow & A(\bigoplus_{y \in V^1(X)} \underline{K}_0(\mathbb{C}(y))) & \longrightarrow 0 \\ & \downarrow T_{-1} & & \downarrow T_0 & & \downarrow T_1 = ? & \\ 0 \longrightarrow & \mathcal{O}_X & \longrightarrow & \underline{H}^0_x(\mathcal{O}_X) & \longrightarrow & \bigoplus_{y \in V^1(X)} \underline{H}^1_y(\mathcal{O}_X) & \longrightarrow 0 \end{array}$$

(define arc spaces here)

Roughly speaking, the vertical maps T_{-1} and T_0 are the tangent maps arising from the functor-theoretic definition of the tangent space to K_1. More precisely, we have the definition

$$TK_1(\mathcal{O}_X) := ker[\phi : K_1(\mathcal{O}_X \otimes D) \to K_1(\mathcal{O}_X)]$$

(where $D = \mathbb{C}[t]/(t^2)$ is the ring of dual numbers). We have

$$K^1(\mathcal{O}_X \otimes D) = (\mathcal{O}_X \otimes D)^* = \{f_0 + f_1 t | f_0 \in \mathcal{O}_X^*, f_1 \in \mathcal{O}_X\}$$

and the map ϕ is given by

$$(f_0 + f_1 t) \mapsto f$$

The kernel is evidently

$$ker[\phi] = \{1 + f_1 t | f_1 \in \mathcal{O}_X\} \cong \mathcal{O}_X$$

where the isomorphism is

$$1 + f_1 t \mapsto f_1$$

The "arcs through the origin in $K_1(\mathcal{O}_X)$" are roughly defined to be

$$A_0(K_1(\mathcal{O}_X)) = \{1 + f_1 t + f_2 t^2 + ...\}$$

Given an arc through the origin, we may truncate it by working mod t^2, thus obtaining an element $1 + f_1 t$ of $TK_1(\mathcal{O}_X)$ which we send to its isomorphic image f_1 in $\mathcal{O}_X$. Finally, given any arc

$$f_0 + f_1 t + f_2 t^2 + ... \in A(K_1(\mathcal{O}_X))$$

we may first "move it to an arc through the origin by the group action of K_1"; i.e. we may send it to

$$1 + \frac{f_1}{f_0} t + \frac{f_2}{f_0} t^2 + ... \in A_0(K_1(\mathcal{O}_X))$$

then truncate it and apply the above map. All this amounts to the definition

$$T_{-1}(f_0 + f_1 t + f_2 t^2 + ...) = \frac{f_1}{f_0}$$

Similarly, given

$$f_0 + f_1 t + f_2 t^2 + \ldots \in A(K_1(\mathbb{C}_X))$$

we may define

$$T_0(f_0 + f_1 t + f_2 t^2 + \ldots) = \frac{f_1}{f_0} \in \mathbb{C}(X)$$

We wish to complete the diagram by defining T_1. We do so by insisting on commutativity of the square

$$\begin{array}{ccc} A(\underline{K}_1(\mathbb{C}(X))) & \xrightarrow{div} & A(\bigoplus_{y \in V^1(X)} \underline{K}_0(\mathbb{C}(y))) \\ \downarrow{\scriptstyle T_0} & & \downarrow{\scriptstyle T_1} \\ \underline{H}^0_x(\mathcal{O}_X) & \xrightarrow{C_1} & \bigoplus_{y \in V^1(X)} \underline{H}^1_y(\mathcal{O}_X) \end{array}$$

Given

$$f(t) = f_0 + f_1 t + f_2 t^2 + \ldots \in A(K_1(\mathbb{C}_X))$$

with f_0 irreducible, the square becomes

$$\begin{array}{ccc} f(t) & \xmapsto{div} & div(f) \\ \downarrow{\scriptstyle T_0} & & \downarrow{\scriptstyle T_1} \\ \frac{f_1}{f_0} & \xmapsto{C_1} & [f_0, f_1] \end{array}$$

So we define

$$T_1(div(f)) = [f_0, f_1]$$

(reducible case)

9.2 Higher-order cases

(f_0 irreducible)

Let B_n be the local artinian $\mathbb{C}$-algebra $B_n := \mathbb{C}(X)[t]/(t^{n+1})$ with maximal ideal $m_n = tB_n$ and residue field $\mathbb{C}$.

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(K_1(\mathcal{O}_X)) & \longrightarrow & A(\underline{K}_1(\mathbb{C}(X))) & \longrightarrow & A(\bigoplus_{y \in V^1(X)} \underline{K}_0(\mathbb{C}(y))) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T^n_{-1}} & & \downarrow{\scriptstyle T^n_0} & & \downarrow{\scriptstyle T^n_1 = ?} & & \\ 0 & \longrightarrow & \mathcal{O}_X \otimes m_n & \xrightarrow{j \otimes Id} & \underline{H}^0_x(\mathcal{O}_X) \otimes m_n & \xrightarrow{C_1 \otimes Id} & \bigoplus_{y \in V^1(X)} \underline{H}^1_y(\mathcal{O}_X) \otimes m_n & \longrightarrow & 0 \end{array}$$

The maps T^n_{-1} and T^n_0 come from extending the tangent maps to higher order using the logarithm. They are given by

$$f = f_0 + f_1 t + f_2 t^2 + \dots \mapsto log(\frac{f(t)}{f_0})$$

where the right hand side is understood in terms of the power series expansion for $log(1+a)$ with $a = \frac{f_1}{f_0}t + \frac{f_2}{f_0}t^2 + \dots = \frac{f(t)}{f_0} - 1$ truncated by setting $t^{n+1} = 0$.

We now focus on the square

$$\begin{array}{ccc} A(\underline{K}_1(\mathbb{C}(X))) & \longrightarrow & A(\bigoplus_{y \in V^1(X)} \underline{K}_0(\mathbb{C}(y))) \\ \big\downarrow T^n_0 & & \big\downarrow T^n_1 = ? \\ \underline{H}^0_x(\mathcal{O}_X) \otimes m_n & \xrightarrow{C_1 \otimes Id} & \bigoplus_{y \in V^1(X)} \underline{H}^1_y(\mathcal{O}_X) \otimes m_n \end{array}$$

and define T^n_1 in such a way as to make it commute.

Consider the map $C_1 \otimes Id$. If we write

$$log(\frac{f(t)}{f_0}) = A_1 t + \dots + A_n t^n$$

mod t^{n+1}, it suffices to give the image of $A_k t^k$ for each k because $C_1 \otimes Id$ sends the t^k-part of $\underline{H}^0_x(\mathcal{O}_X) \otimes m_n$ to the t^k-part of $\bigoplus_{y \in V^1(X)} \underline{H}^1_y(\mathcal{O}_X) \otimes m_n$. The denominator of A_k is f_0^k, where f_0 is by hypothesis irreducible and relatively prime to the numerator. Therefore, $C_1 \otimes Id$ sends $A_k t$ to $[f_0^k, f_0^k A_k]$. Now

$$f_0^k A_k = \frac{1}{k!}\frac{d^k}{dt^k}(f_0^k log(\frac{f(t)}{f_0}))|_{t=0}$$

So we have

$$T^n_1(div(f(t)) = \sum_{k=1}^{n}[f_0^k, \frac{1}{k!}\frac{d^k}{dt^k}(f_0^k log(\frac{f(t)}{f_0}))|_{t=0}]t^k$$

(now do reducible case)

10 Tangent Spaces at Effective Zero-Cycles

First consider the tangent space to $Z^1(X)^+$ at $m\{0\}$. We have

$$z(t) = div(f(t))$$

where $f(t)$ is well defined up to units in the local ring. Also

$$z(0) = div(f(0)) = m\{0\}$$

so that $f_0 = f(0) = x^m$ up to units. We have the pairing

$$(f(t), \omega) \mapsto -Res_0 \frac{f_1}{f_0} \omega$$

.

Note that multiplying $f(t)$ by a unit multiplies both f_0 and f_1 by the same unit, which cancels out on the right hand side. So we may assume that $f_0 = x^m$. Thus, we may write

$$\frac{f_1}{f_0} = \frac{a_m}{x^m} + \ldots + \frac{a_1}{x} + h$$

where h is holomorphic.

Since ω is also holomorphic, the right hand side depends only on the principal part $[\frac{f_1}{f_0}] := \frac{a_m}{x^m} + \ldots + \frac{a_1}{x}$ of $\frac{f_1}{f_0}$. Let $\mathcal{P}_{X,0,m}$ be the vector space of principal parts of the form

$$\frac{a_n}{x^m} + \ldots + \frac{a_1}{x}$$

i.e. principal parts at 0 with poles of at most order m. We may now refine the above pairing to

$$\mathcal{P}_{X,0,m} \otimes \Omega^1_{X/\mathbb{C},0} \to \mathbb{C}$$
$$\tau \otimes \omega \mapsto Res_0 \tau \omega$$

Now since τ has a pole of at most order m at 0, there will be no residue if $\omega \in m_0^m \Omega^1_{X/\mathbb{C},0}$. Therefore we may further refine the pairing to

$$\mathcal{P}_{X,0,m} \otimes (\Omega^1_{X/\mathbb{C},0} / m_0^m \Omega^1_{X/\mathbb{C},0}) \to \mathbb{C}$$

with the same formula. This last pairing is easily seen to be non-degenerate. This allows us to identify the tangent space:

$$T_{m\{0\}} Z^1(X)^+ \cong Hom_{\mathbb{C}}(\Omega^1_{X/\mathbb{C},0} / m_0^m \Omega^1_{X/\mathbb{C},0}, \mathbb{C})$$

and similarly

$$T_{mx}Z^1(X)^+ \cong Hom_{\mathbb{C}}(\Omega^1_{X/\mathbb{C},x}/m_x^m\Omega^1_{X/\mathbb{C},x}, \mathbb{C})$$

for any $x \in X$.

Moving on to the tangent space at $\sum_{i=1}^m n_i x_i$, we now have the pairing

$$(f,\omega) \mapsto -\sum_{i=1}^m Res_{x_i}\frac{f_{i1}}{f_{i0}}\omega$$

Arguing as we did in the case of a single multiple point, we see that only the principal parts $[\frac{f_{i1}}{f_{i0}}]$ matter, and that there will be no residues if $\omega \in \cap_{i=1}^m m_{x_i}^{n_i}\Omega^1_{X/\mathbb{C},x_i}$. Therefore we may refine the pairing to

$$(\prod_{i=1}^m \mathcal{P}_{X,x_i,n_i}) \otimes_{\mathbb{C}} (\cap_{i=1}^m \Omega^1_{X/\mathbb{C},x_i}/\cap_{i=1}^m m_{x_i}^{n_i}\Omega^1_{X/\mathbb{C},x_i}) \to \mathbb{C}$$

This pairing is easily seen to be non-degenerate, so

$$T_{\sum_{i=1}^m n_i x_i}Z^1(X)^+ \cong Hom_{\mathbb{C}}(\cap_{i=1}^m \Omega^1_{X/\mathbb{C},x_i}/\cap_{i=1}^m m_{x_i}^{n_i}\Omega^1_{X/\mathbb{C},x_i}, \mathbb{C})$$

By using the description

$$T_{\sum_{i=1}^m n_i x_i}Z^1(X)^+ = \prod_{i=1}^m \mathcal{P}_{X,x_i,n_i}$$

we see that with this definition $dim_{\mathbb{C}}T_{\sum_{i=1}^m n_i x_i}Z^1(X)^+ = \sum_{i=1}^m n_i$, which is the dimension of the corresponding symmetric product (in the case of curves, the symmetric product is nonsingular, so this is what we should expect).

The situation is far less clear when considering the tangent space at the same point in $Z^1(X)$, because there are many more arcs (involving creation/annihilation) beginning at this point (this also raises questions about the topology).

11 Jet Spaces at Effective Zero-Cycles

Definition 8 *The nth order jet space to $Z^1(X)^+$ at $\sum_{i=1}^m n_i x_i$, denoted $T^n_{\sum_{i=1}^m n_i x_i}Z^1(X)^+$, is the set of all equivalence classes of arcs originating at $\sum_{i=1}^m n_i x_i$ under nth order equivalence, where $z_1(t)$ and $z_2(t)$ are equivalent to nth order if and only if $I_k(z_1(t),\omega) = I_k(z_1(t),\omega)$ for all $k = 1,...,n$.*

For simplicity, begin with the case of arcs through mx.

Note that

$$-\sum_{j=1}^{n} Res_x A_j \omega t^j = 0$$

if and only if

$$-Res_x A_j \omega = 0$$

for all $j = 1, ..., n$

Thus, we consider the pairings

$$(f, \omega) \mapsto -\sum_{j=1}^{n} Res_x A_j \omega t^j$$

Now f is specified by giving the polynomials $f_0, f_1, ...$ Since the exponential and logarithmic maps are inverses of each other, we may also specify f by giving the rational functions $A_1, A_2, ...$ Clearly only the principal parts of the A_i matter in calculating the residues, and we can check directly that A_i has a pole of at worst order im at x. Because of this, forms in $m_x^{nm}\Omega^1_{X/\mathbb{C},x}$ are annihilated, so we have a pairing

$$(\prod_{j=1}^{n} \mathcal{P}_{X,x,jm}) \otimes_{\mathbb{C}} (\Omega^1_{X/\mathbb{C},x}/m_x^{nm}\Omega^1_{X/\mathbb{C},x}) \to t\mathbb{C}[t]/t^{n+1}$$

If this pairing is nondegenerate (?), then (WRONG FORMULA; FIX)

$$T^n_{mx} Z^1(X)^+ \cong Hom_{\mathbb{C}}(\Omega^1_{X/\mathbb{C},x}/m_x^{nm}\Omega^1_{X/\mathbb{C},x}, t\mathbb{C}[t]/t^{n+1})$$

To be continued...

12 Principal Parts, local cohomology, and the Ext-definition of the tangent sheaf

Fact 4 $\mathcal{P}_{X,x} \cong lim_{n\to\infty}\mathcal{E}xt^1_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/m_x^n, \mathcal{O}_{X,x}) \cong H^1_{m_x}(\mathcal{O}_{X,x}) \cong H^1_x(\mathcal{O}_X)$

Proof: □

13 The tangent space to $CH^1(X)$

In the curve case, the formal tangent space to $CH^1(X)$ is

$$H^1(X, \mathcal{O}_X)$$

while the geometric tangent space is

$$TZ^1(X)/TZ^1_{rat}(X).$$

Notation and Discussion:

We have the standard principal parts sequence (PPS) and divisor sequence (DS):

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{R}_X \longrightarrow \mathcal{P}_X \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{R}_X^* \longrightarrow \underline{Z}^1(X) \longrightarrow 0$$

$\mathcal{O}_X$ is tangent to $\mathcal{O}_X^*$ wrt the functor K_1 in the sense that

$$K_1(\mathcal{O}_X) = \mathcal{O}_X^*$$

and

$$TK_1(\mathcal{O}_X) = \mathcal{O}_X)$$

and similarly for $\mathcal{R}_X$ and $\mathcal{R}_X^*$.

We agreed to tentatively define $K_1(\mathcal{P}_X) := \underline{Z}^1(X)$. With this definition we have previously shown (in the notes I wrote up over the summer) that we can make rigorous sense of the statements

$$K_1(PPS) = DS$$

$$TK_1(PPS) = PPS$$

However, this is probably the wrong point of view; noting that

$$\underline{Z}^1(X) = \oplus_{x \in X} \underline{\mathbb{Z}}_x = \oplus_{x \in X} K_0(k(x))$$

as expressed in the general form of the CFR, we see that $\underline{Z}^1(X)$ really has nothing to do with applying K_1 to $K_1(\mathcal{P}_X)$; it really involves K_0. We can get away with viewing it in terms of K_1

because the resolution is a short exact sequence: this allows us to (rather crudely) define K_1 of the quotient $\mathcal{P}_X = \frac{\mathcal{R}_X}{\mathcal{O}_X}$ to be the quotient $\frac{K_1(\mathcal{R}_X)}{K_1(\mathcal{R}_X)}$. This is an ugly ad hoc definition which provides no insight and does not seem to generalize to higher dimensions.

However, this definition serves the present purpose of expressing the PPS as the tangent sequence to the DS, which is all I need for what I want to illustrate.

$$Z^1(X) = H^0(X, \underline{\mathcal{Z}}^1(X))$$

$= H^0(X, \mathbb{G}_m(\mathcal{P}_X))$ so $TZ^1(X)$ means $TH^0(X, \mathbb{G}_m(\mathcal{P}_X)) = H^0(X, T\mathbb{G}_m(\mathcal{P}_X))$ (since H^0 is middle-exact) $= H^0(X, \mathcal{P}_X)$ (since $T\mathbb{G}_m$ is the forgetful functor) $= H^0(X, T\mathcal{Z}^1_X)$. We write $\mathcal{P}_X = T\mathcal{Z}^1_X$ with the understanding that the tangent is being taken with respect to the functor $\mathbb{G}_m$. Strictly speaking, $T\mathcal{Z}^1_X$ and $TZ^1(X)$ are examples of bad notation because tangent objects have no meaning except with respect to a specified functor. In the case of $T\mathcal{Z}^1_X$, the functor is $\mathbb{G}_m$, and in the case of $Z^1(X)$, the functor is the composite $H^0(X, \mathbb{G}_m(-))$.

Further,

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{R}_X) \longrightarrow H^0(X, \mathcal{P}_X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

is the tangent sequence to

$$1 \longrightarrow H^0(X, \mathcal{O}^*_X) \longrightarrow H^0(X, \mathcal{R}^*_X) \longrightarrow H^0(X, \mathcal{Z}^1_X) \longrightarrow H^1(X, \mathcal{O}^*_X) \longrightarrow 1$$

in the following sense:

(explain, using the δ-functor $H^*(X, -)$ and the functor $\mathbb{G}_m$.)

The sequences above can be rewritten as

$$0 \longrightarrow \mathcal{O}(X) \longrightarrow \mathcal{R}(X) \longrightarrow TZ^1(X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

and

$$1 \longrightarrow \mathcal{O}(X)^* \longrightarrow \mathcal{R}(X)^* \longrightarrow \mathcal{Z}^1(X) \longrightarrow Pic(X) \longrightarrow 1$$

$$CH^1(X)$$

By definition, $Z^1_{rat}(X) \subset Z^1(X)$ is the image of $\mathcal{R}(X)^*$ in $Z^1(X)$; i.e. the global 0-cycles rationally equivalent to zero. Correspondingly, $TZ^1_{rat}(X) \subset TZ^1(X) = \mathcal{P}(X)$ is the image of $\mathcal{R}(X)$ in $TZ^1(X)$.

Issue: In what sense is the single tangent object $TZ^1(X)$ of $Z^1(X)$ related to the tangent spaces at the various points of $Z^1(X)$, and what justification is there for calling this formulation "geometric?"

Because the sheaf $\mathcal{P}_X$ is flasque, there are surjective maps

$$\mathcal{P}(X) \to \mathcal{P}_{X,x}$$

for every $x \in X$, so in the obvious sense $\mathcal{P}(X)$ "contains all the information" about the tangent spaces at the various points of $Z^1(X)$. (need to elaborate further)

Slightly different point of view:

The sheaf $\mathcal{O}_X$ has flasque resolution

$0 \longrightarrow \mathcal{O}_X \xrightarrow{i} \mathcal{R}_X \xrightarrow{j} \mathcal{P}_X \longrightarrow 0$

We can use this to compute sheaf cohomology; the sheaf cohomology of $\mathcal{O}_X$ is the cohomology of the complex

$\Gamma(X, \mathcal{R}_X) \xrightarrow{j} \Gamma(X, \mathcal{P}_X) \longrightarrow 0$

We get

$$H^0(X, \mathcal{O}_X) = ker(j) \cong \Gamma(X, \mathcal{O}_X)$$

because Γ is left-exact, while

$$H^1(X, \mathcal{O}_X) = \Gamma(X, \mathcal{P}_X)/im(j)$$

This is the same result from a different angle. (elaborate)

Move to surfaces:

Formal tangent space to $CH^2(X)$ is $H^2(X, \Omega^1_{X/\mathbb{Q}})$ (Bloch and Van der Kallen)

In analogy to the curve case, the geometric tangent space is a quotient of $TZ^2(X)$ ([GG] p 141) where

$$TZ^2(X) := H^0(X, \oplus_{x \in X} H^2_x(\Omega^1_{X/\mathbb{Q}}))$$

We want to extend the analogy by means of flasque resolutions and tangent sequences.

According to [GG], the Cousin flasque resolution

$0 \longrightarrow \Omega^1_{X/\mathbb{Q}} \longrightarrow \underline{\Omega}^1_{\mathbb{C}(X)/\mathbb{Q}} \longrightarrow \oplus_Y \underline{H}^1_y(\Omega^1_{X/\mathbb{Q}}) \longrightarrow \oplus_x \underline{H}^2_x(\Omega^1_{X/\mathbb{Q}}) \longrightarrow 0$

is the tangent sequence to the BGQ sequence

$0 \longrightarrow K_2(\mathcal{O}_X) \longrightarrow \underline{K}_2(\mathbb{C}(X)) \longrightarrow \oplus_Y \underline{\mathbb{C}}(Y)^* \longrightarrow \oplus_x \underline{\mathbb{Z}}_x \longrightarrow 0$

[GG] put out a great deal of effort to justify this assertion, but in a rather ad hoc way. Presumably because they are interested in giving geometric significance to formal results, they often focus on concrete situations (for example, on pg 126-? they discuss explicitly how to calculate the tangent to an arc in $\oplus_Y \underline{\mathbb{C}}(Y)^*$ when its equations are known). What remains unclear (at least to me) is the formal picture itself.

There are two different concepts of tangent space or tangent object to consider. One is the formal tangent space with respect to a functor, which is a *kernel*; for instance

$$TK_2(\mathcal{O}_X) := ker[K_2(\mathcal{O}_X \otimes D) \to K_2(\mathcal{O}_X)] \cong \Omega^1_{X/\mathbb{Q}}$$

(where the isomorphism comes from Van der Kallen's theorem). This justifies the statement that the CFR is tangent to the BGQS in the first term.

The second concept of tangent space or tangent object is equivalence classes of arcs in a space, which is a *cokernel*; this is the point of view in elementary differential geometry and the point of view in [GG]'s many concrete examples.

What I would like to know is if the remaining terms of the CFR are tangent to the remaining terms of the BGQS in the first, formal sense, and if so, what the functors are. For example, for what functor F is

$$TF(\oplus_Y \underline{\mathbb{C}}(Y)^*) = \oplus_Y \underline{H}^1_y(\Omega^1_{X/\mathbb{Q}})$$

?

The most naiive guess would be that the Cousin flasque resolution of $\Omega^1_{X/\mathbb{Q}}$ comes from applying the tangent functor TK_2 to the Cousin flasque resolution of $\mathcal{O}_X$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{R}_X \longrightarrow \oplus_Y \underline{H}^1_y(\mathcal{O}_X) \longrightarrow \oplus_x \underline{H}^2_x(\mathcal{O}_X) \longrightarrow 0$$

But this makes no sense; for instance, it makes no sense to apply K_2 to $\oplus_Y \underline{\mathbb{C}}(Y)^*$.

14 The formal completion to $CH^n(X)$

Discuss $n = 2$ case.

Bloch:

$$CH^2(X) = H^2(X, K_2(\mathcal{O}_X))$$

Notation:

$$\mathcal{K}_{2,X} := K_2(\mathcal{O}_X)$$

Formal completion:

$$\hat{CH}^2(X)(A) := ker[H^2(X_A, \mathcal{K}_{2,X_A}) \to H^2(X, \mathcal{K}_{2,X})]$$

Because of splitting have

$$\hat{CH}^2(X)(A) = H^2(X, \hat{\mathcal{K}}_{2,X}(A))$$

where

$$\hat{\mathcal{K}}_{2,X}(A) := ker[\mathcal{K}_{2,X_A} \to \mathcal{K}_{2,X}]$$

So interested in $\hat{\mathcal{K}}_{2,X}(-)$ as a functor {loc. art. k-alg} → {ab gp}

Steinstra general formula:

$$\hat{\mathcal{K}}_{2,X}(A) = \frac{\Omega^1_{X\otimes A, X\otimes m}}{d(\mathcal{O}_X \otimes_k m)}$$

Hesselholt special case (at ring level):

$$ker[K_{n+1}(S \otimes_k A_N) \to K_{n+1}(S)] \cong \oplus_{p\geq 0}(\Omega^{n-2p}_{S/\mathbb{Q}})^{N-1}$$

where $A_N := k[\epsilon]/(\epsilon)^N$

Move to sheaf level, set $n+1=2$, use Steinstra's notation:

$$\hat{\mathcal{K}}_{2,X}(A_N) = (\Omega^1_{X/\mathbb{Q}})^{N-1}$$

Thus

$$\hat{CH}^2(X)(A_N) = H^2(X, \hat{\mathcal{K}}_{2,X}(A_N)) = H^2(X, (\Omega^1_{X/\mathbb{Q}})^{N-1})$$

Can compute sheaf cohom of $(\Omega^1_{X/\mathbb{Q}})^{N-1}$ from a flasque resolution; take $(N-1)$-fold direct sum of CFR; get

$$0 \longrightarrow (\Omega^1_{X/\mathbb{Q}})^{N-1} \longrightarrow (\underline{\Omega}^1_{\mathbb{C}(X)/\mathbb{Q}})^{N-1} \xrightarrow{i_0} (\oplus_Y \underline{H}^1_y(\Omega^1_{X/\mathbb{Q}}))^{N-1} \xrightarrow{i_1} (\oplus_x \underline{H}^2_x(\Omega^1_{X/\mathbb{Q}}))^{N-1} \longrightarrow 0$$

Accordingly

$$H^2(X, (\Omega^1_{X/\mathbb{Q}})^{N-1}) \cong \frac{\Gamma(X, (\oplus_x \underline{H}^2_x(\Omega^1_{X/\mathbb{Q}}))^{N-1})}{i_1(\Gamma(X, (\oplus_Y \underline{H}^1_y(\Omega^1_{X/\mathbb{Q}}))^{N-1}))}$$

In the case $N=2$, $A_N = D$ we get the tangent space

$$TCH^2(X) = H^2(X, \Omega^1_{X/\mathbb{Q}}) \cong \frac{\Gamma(X, \oplus_x \underline{H}^2_x(\Omega^1_{X/\mathbb{Q}}))}{i_1(\Gamma(X, \oplus_Y \underline{H}^1_y(\Omega^1_{X/\mathbb{Q}})))}$$

[GG] defined (p 141)

$$TZ^2(X) := \Gamma(X, \oplus_x \underline{H}_x^2(\Omega^1_{X/\mathbb{Q}}))$$

and

$$TZ^1_1(X) := \Gamma(X, \oplus_Y \underline{H}_y^1(\Omega^1_{X/\mathbb{Q}}))$$

With this notation, we have

$$TCH^2(X) = \frac{TZ^2(X)}{i_1(TZ^1_1(X))}$$

which represents $TCH^2(X)$ as "the tangent space to $Z^2(X)$ modulo tangents to rational equivalences."

Here $Z^2(X)$ and $Z^1_1(X)$ are the global sections of the corresponding objects from the BGQ sequence for $\mathcal{O}_X$:

$$Z^2(X) = \Gamma(X, \oplus_x \underline{\mathbb{Z}}_x)$$

$$Z^1_1(X) = \Gamma(X, \oplus_Y \underline{\mathbb{C}}(Y)^*)$$

([GG] omit the global sections functor on page 141, identifying Z^2 and Z^1_1 as sheaves, but this must be an oversight.)

Discuss mappings: what is crucial is that the map

$$i_1 : TZ^1_1(X) \to TZ^2(X)$$

coming from the CFR is the "differential" of the map

$$Z^1_1(X) \to Z^2(X)$$

coming from the BGQ sequence.

Discussion: Observe why the CFR in particular yields the nice geometric picture of tangents modulo tangents to rational equivalences. Any Γ-acyclic resolution of $\Omega^1_{X/\mathbb{Q}}$ will yield the desired cohomology $H^2(X, \Omega^1_{X/\mathbb{Q}})$, but what is special about the CFR is that it is the "tangent sequence" to the BGQ, whose final term is the sheaf of 0-cycles $\oplus_{x \in X} \underline{Z}_x$, with global sections $Z^2(X)$. Thus the CFR gives $H^2(X, \Omega^1_{X/\mathbb{Q}})$ as a quotient of $TZ^2(X)$. What remains the crucial mystery is the exact nature of the "tangent operation" that transforms the BGQ into the CFR. When is it defined? How general is it?

Discussion of Quillen's proof of Bloch's formula and the desirability of rigorously defining the operation of computing a "tangent sequence."

Bloch's formula:

$$CH^p(X) = H^p(X, \mathcal{K}_{p,X})$$

BGQ complex:

$$0 \longrightarrow K'_p(X) \longrightarrow \coprod_{x \in X_0} K_p(k(x)) \longrightarrow \coprod_{x \in X_1} K_{p-1}(k(x)) \longrightarrow \dots$$
$$\dots \longrightarrow \coprod_{x \in X_p} K_0(k(x)) \longrightarrow 0$$

(Distinguish between K' and K)

I think this exists for any appropriate X and p, not just $p = n = dim(X)$.

Note that the final term is $\coprod_{x \in X_p} K_0(k(x)) = \coprod_{x \in X_p} \underline{Z}_x$ whose group of global sections is $Z^p(X)$.

The online notes by Eric Friedlander state that Quillen's method of proof of Bloch's formula (find Quillen's original proof) was to show that that the BGQ complex is a flasque resolution of $K'_p(X)$ under (sufficiently general) conditions and that the image of the "final differential" d_1

$$\coprod_{x \in X_{p-1}} K_1(k(x)) \xrightarrow{d_1} \coprod_{x \in X_p} K_0(k(x)) \longrightarrow 0$$

is precisely Z^p_{rat} the codimension-p cycles rationally equivalent to zero (I am sure he actually means that Z^p_{rat} is the image of the map on global sections induced by d_1). Therefore we can compute the pth sheaf cohomology of $K'_p(X)$ $(= \mathcal{K}_{p,X}?)$ by means of this resolution as follows:

$$H^p(X, K'_p(X)) = H^p(\Gamma(GC)) = \frac{H^0(X, \coprod_{x \in X_p} K_0(k(x)))}{d_1(H^0(X, \coprod_{x \in X_{p-1}} K_1(k(x))))}$$
$$= \frac{Z^p(X)}{Z^p_{rat}(X)} = CH^p(X)$$

Now suppose we had a well-defined "tangent operation" that we could perform on the GC. Presumably the first-term in the tangent sequence would be $TK'_p(X) = \Omega^{p-1}_{X/\mathbb{Q}}$ (although does this contradict Hesselholt's result?). Crucial question: is the tangent sequence (if it exists) a flasque resolution of $\Omega^{p-1}_{X/\mathbb{Q}}$? If so, then we can use it to compute the sheaf cohomology of $\Omega^{p-1}_{X/\mathbb{Q}}$. If we follow [GG] and define

$$\underline{Z}^n_m(X) := \coprod_{x \in X_n} K_m(k(x))$$

and

$$Z_m^n(X) = H^0(X, \underline{Z}_m^n(X))$$

then the last two terms in this resolution (after taking global sections) would be

$TZ_1^{p-1}(X) \xrightarrow{d_1 *} TZ_0^p(X) \longrightarrow 0$

where d_1^* is the differential of d_1, and we would have

$$T_{formal}CH^p(X) = H^p(X, T(GC)) = \frac{TZ^p(X)}{d_1^*(TZ_1^{p-1}(X))}$$

in general.

Of course, there would still be the issue of defining tangent maps from arc spaces, etc.